SUPERCritical AND SUBCRITICAL LONG'S VORTICES

Ramón FERNÁNDEZ-FERIA
E. T. S. Ingenieros Industriales
Universidad de Málaga, 29013 Malaga, SPAIN

ABSTRACT
A spatial linear stability analysis of Long's vortex shows that, although both Type I and Type II vortices are convectively unstable for counter-rotating spiral modes, only Type II Long's vortices with negative axial velocity at the axis can in addition sustain unstable spiral modes with negative group velocities. Thus, Type I and Type II Long's vortices are supercritical and subcritical swirling flows in Benjamin's sense, respectively. The transition between the two types of flows taking place when the axial velocity at the axis becomes zero.

INTRODUCTION
Long's (1961) vortex has been extensively considered as a simple model for high Reynolds number vortices of geophysical and engineering interest, mainly because it is an exact solution to the near-axis approximation of the Navier-Stokes equation which is non-parallel and consistently includes a relatively important axial flow, both characteristics present in most real vortices of interest. In particular, its stability has been analysed by a number of authors using different techniques and degrees of approximation, with the main objective of trying to elucidate and predict some of the interesting properties that highly swirling flows present in practice. Most of these previous works considered the temporal stability (i.e. with given real wave number and unknown complex frequency) of Long's vortex using a parallel flow approximation (see, e.g., Fernández-Feria (1996) for the most significant references: in that work, hereafter referred to as T, the effect of viscosity and of the non-parallelism of the basic flow is also taken into account in the temporal stability analysis of Long's vortex). In the present work, the results of a spatial stability analysis (i.e. with given real frequency look for the complex axial wave number) of Long's vortex which also takes into account the effect of viscosity and the non-parallelism of the basic flow (locally) are reported. The spatial analysis is more appropriate to study the evolution of waves as they propagate from a given forced oscillation at a given location, which is the situation usually met in experiments. It is shown that the local spatial analysis reproduces the results of the temporal instability calculations when the group velocity is positive (convective instabilities). However, new unstable counter-rotating spiral modes with negative group velocities are found here using the spatial formulation for Type II vortices, not found with the temporal analysis. As discussed in the last section, these new unstable modes establish a fundamental difference between Type I and Type II Long's vortices.

FORMULATION OF THE PROBLEM
The basic vortex
Long's vortex is a similarity solution to the near-axis boundary layer approximation of the steady, incompressible and axisymmetric Navier-Stokes equations, matching an inviscid flow with velocity and pressure fields inversely proportional to the distance $r$ to the axis of symmetry. Long (1961) showed that there are two solutions for $M > M^*$ (termed as Type I and Type II solutions by Burggraf and Foster (1977)), and none for $M < M^*$, where $M$ is the dimensionless flow force and $M^* \approx 3.75$ is a critical value. In cylindrical polar co-ordinates $(r, \theta, z)$, the vortex has the self-similar structure

$$\Psi = \nu z f(\xi), \quad (1)$$

$$V = \frac{\nu z}{\delta^2(\xi)} \gamma(\xi), \quad \frac{P}{\rho} = \frac{(\nu z)^2}{\delta^2(\xi)} \beta(\xi), \quad (2)$$

where $\Psi$ is the stream function for the meridional motion, through which the axial and radial velocity components, $W$ and $U$, are:

$$W = \frac{\nu z}{\delta^2} 2f'(\xi), \quad U = -\frac{\nu}{r} [f(\xi) - 2z f'(\xi)] ; \quad (3)$$

$V$ is the azimuthal velocity component, $\nu$ the kinematic viscosity, and $\delta(z)$ the vortex core thickness.

$$\delta(z) = \frac{\nu z}{W_0}, \quad (4)$$
being a constant with the same dimensions as \( v \) characterizing the external inviscid flow: the similarity variable \( \xi \) is defined by

\[
\xi \equiv \eta^2, \quad \eta = \frac{r}{\delta(x)}.
\]  

(5)

The functions \( f, \gamma \) and \( \beta \) are governed by a set of three non-linear ordinary differential equations (see T). An unique solution exists for each value of the non-dimensional axial velocity at the axis, \( A_1 \equiv f'(0) \), in the allowed interval \(-1/\sqrt{2} < A_1 < \infty \). Type I solutions have a positive axial velocity at the axis, with \( A_1 \) within the interval \((A_1^*, \infty)\), where \( A_1^* \approx 0.15 \) corresponds to the critical (or folding) value of \( M \). While most Type II solutions have a negative axial velocity at the axis, with \( A_1 \) in the interval \((-1/\sqrt{2}, A_1^*)\) (see Fig. 1). In the limit of large \( A_1 \) ( \( M \) large), Type I solution corresponds to an intense swirling jet with large positive axial velocity at the axis, while for \( A_1 \rightarrow -1/\sqrt{2} \) (again \( M \) large), Type II solution has the form of a ring-jet with large positive axial flow on the ring and negative axial flow in its interior (see Foster and Smith (1989) for asymptotic solutions in these two limits).

Figure 1: (a) Function \( M(A_1) \). (b) Axial velocity profiles for three different values of \( A_1 \).

Stability formulation and numerical method.

The flow variables \((u, v, w)\) and \( p \) are decomposed as usual, into a mean part \((\bar{U}, \bar{V}, \bar{W})\) and \( P \), and a small perturbation. After (2)-(3),

\[
\omega = \frac{\nu Z}{\delta^3} [2f' + \bar{w}] \quad u = \frac{\nu}{r} [-f + 2\xi f' + \frac{r Z}{\delta^2} \bar{w}] 
\]

(6)

\[
v = \frac{\nu Z}{\delta^3} [\tau + \bar{r}] \quad \frac{p}{\rho} = \frac{(\nu Z)^2}{\delta^4} [\beta + \bar{p}],
\]

(7)

where the perturbations are written in the standard form

\[
s \equiv [\bar{u}, \bar{v}, \bar{w}, \bar{p}]^T \equiv S(x, \xi) \chi(x, \theta, t), \quad \chi \equiv \frac{x}{z_o},
\]

(8)

with the complex amplitude,

\[
S(x, \xi) = [iF(x, \xi), G(x, \xi), H(x, \xi), \Pi(x, \xi)]^T.
\]

(9)

and the exponential part describing the wave-like nature of the disturbances,

\[
\chi(x, \theta, t) = \exp \left[ \frac{1}{\Delta \omega} \int_{x_o}^{x} a(x') dx' + i(n\theta - \Omega t) \right].
\]

(10)

The use of an axial scale-length \( z_o \) in addition to the radial characteristic length \( \delta \), defined as the vortex thickness at \( z_o \), \( \delta = \nu z_o / \Delta \omega \), is assumed to be small within the present near-axis boundary layer approximation (note that terms \( O(\Delta^2) \) are neglected in the derivation of Long's vortex). The non-dimensional order of unity, axial wavenumber \( a \) is defined as \( \delta \), times the dimensional wave number \( k \):

\[
a(x) \equiv \delta k(x) \equiv \gamma(x) + ia(x),
\]

(11)

which accounts for the fast, wave-like variation of the perturbations. Its real part \( \gamma(x) \) is the exponential growth rate, and the imaginary part \( a(x) \) is the axial wavenumber. A non-dimensional, order of unity, frequency \( \omega \) is also defined:

\[
\omega = \frac{\Omega z_o^3}{\nu^2}.
\]

(12)

Substituting (6)-(12) into the incompressible Navier-Stokes equations, and neglecting second-order terms in both the small perturbations and the inverse of the local Reynolds number,

\[
Re^{-1} \equiv \Delta(\omega) \equiv \frac{\delta(z)}{\bar{z}} = \Delta \omega \ll 1,
\]

(13)

(note that the local Reynolds number is constant along the axis in Long's vortex), a set of linear parabolized stability equations results for \( S, a \) and \( \omega \). In these equations, the small terms \( O(\Delta) \) take into account three different effects on the stability of the perturbations: the effect of viscosity, the effect of the non-parallelism of the basic flow, and the effect of the streamwise evolution of the perturbations. Here, as in T, this last effect will be neglected, so that the terms proportional to the streamwise derivatives \( \partial S / \partial z \) will disappear from the equations, thus becoming a set of ordinary differential equations instead of partial differential equations (it is shown elsewhere that these terms accounting for the history of the perturbations are relatively important only for low Reynolds numbers when the growth rate is very small). The resulting local stability equations, which include the effect of viscosity and the non-parallelism of the flow partially, may be written as:

\[
[L_{ao} + \Delta L_{o1} + aL_1 + a^2 \Delta L_2] S = 0,
\]

(14)
where $L_{\infty}$, $L_{01}$, $L_1$ and $L_2$ are linear, order of
unity, operators which depend on $x$ and $\xi$ (note
that only the operators $L_{\infty}$ and $L_1$ would appear
in an inviscid stability analysis with parallel flow
approximation). $L_{\infty}$ depends linearly on the fre-
quency $\omega$ and contains $\partial F/\partial \xi$ terms, while $L_{01}$ con-
tains both $\partial F/\partial \xi$ and $\partial^2 F/\partial \xi^2$ terms. This equation
is solved with the following homogeneous radial
boundary conditions:

\begin{align}
    \xi \to \infty & : \quad F = G = H = 0; \quad (15) \\
    \xi = 0 & : \quad F = G = 0, \frac{\partial H}{\partial \xi} = 0, (n = 0), \quad (16) \\
    F \pm G = 0, \frac{\partial F}{\partial \xi} = 0, H = 0, (n = \pm 1). \quad (17) \\
    F = G = H = 0. \quad (|n| > 1). \quad (18)
\end{align}

For the spatial stability analysis considered
here, (14)-(18) constitute a non-linear eigenvalue
problem where, given a real frequency $\omega$, one
looks for the complex eigenvalues $\alpha$ and complex
eigenfunctions $S$. For a given basic vortex ($A_1$),
azimuthal wave number ($n$), and axial location ($x$),
the flow is unstable for the chosen frequency
if the real part of $\alpha$, $\gamma$, is positive. To solve
numerically the problem, the $\xi$-dependence of $S$
is discretized using a staggered Chebyshev spectral
collocation technique developed by Khorrami
(1991). This method has the advantage of
eliminating the need of two artificial pressure
boundary conditions at $\xi = 0$ and $\xi = \infty$, which
for that reason are not included in (15)-(18).
The boundary conditions at infinity (15) are applied
at a truncated radial distance $\xi_{\text{max}}$, chosen large
enough to ensure that the results do not depend
on that truncated distance ($\xi_{\text{max}} = 5000$ was
used in most of the reported computations).

Figure 2: $\gamma(\omega)$ (continuous lines) and $\alpha(\omega)$
(dashed lines) for convectively unstable modes
with $n = -1$, for $x = 1$, $\Delta_0 = 0.001$ and
increasing values of $A_1$. $N = 40$ and 60.

RESULTS AND DISCUSSION

Figure 2 shows the growth rates ($\gamma$) and axial
wavenumbers ($\alpha$) of the most unstable inviscid
($\Delta = 10^{-3}$) modes with positive group velocities
($c_g \equiv \partial \omega / \partial \alpha > 0$) for azimuthal wave-
number $n = -1$ and values of $A_1$ ranging from 0.2 (Type
I vortex near the folding value $A_1^*$) to $-0.6$ (Type
II vortex near its minimum value $-1/\sqrt{2}$). The
maximum growth rate is always about 0.1, with
the range of unstable frequencies increasing as
$A_1$ decreases. These convectively unstable modes
are the most unstable ones for $A_1 > 0$, and
correspond to those obtained with the temporal
stability analysis of $T$. However, as shown in
Fig. 3, the spatial analysis reveals the existence
of highly unstable modes with negative group
velocities for $A_1 < 0$, which do not exist for
$A_1 > 0$. In fact, $c_g \to 0$ as $A_1 \to 0^-$ (see Fig.
4). For $-A_1$ small, both the growth rate and the
wavenumber are very large for low frequencies,
with $c_g(\omega)$ almost constant for all values of $\omega$. As
$A_1$ decreases, $\gamma$ decreases, but remaining always
larger than the corresponding to the convectively
unstable modes of Fig. 2, and $-c_g$ increases.
For $A_1$ very near its minimum value $-1/\sqrt{2}$, a bunch of unstable modes, both with positive and negative $c_g$, appears. The unstable mode for $A_1 = -0.7$ plotted in Fig.3 does not correspond to that with largest $\gamma$, which has $c_g > 0$ and it is neither plotted in Fig.2, but to the one with $c_g < 0$ in some frequency range (note that, $c_g \to -\infty$ for $\omega \approx 0.2$).

**Figure 3:** $\gamma(\omega)$ (continuous lines) and $\alpha(\omega)$ (dashed lines) for modes with $n = -1$ and $c_g < 0$. $x = 1$, $\Delta_0 = 0.001$ and $A_1 = -0.05, -0.1, -0.1, -0.7$ (top to bottom). $N = 100$.

**Figure 4:** Group velocity at $\omega = 0.2$ for the unstable modes of Fig. 3.

These results show that Type II Long's vortices with $A_1 < 0$ can sustain both upstream- and downstream-propagating unstable modes with $n = -1$, so that they are subcritical in Benjamin's (1962) sense. Type I Long's vortices, on the other hand, are supercritical because they can only sustain downstream-propagating, or convective, unstable modes. This fundamental difference between Type I and Type II Long's vortices has to be added to that found in T. where it was shown that only Type II Long's vortices are unstable for axisymmetric ($n = 0$) disturbances. Another interesting result is that, although convectively unstable modes for Type II flows exist for any value of the azimuthal wave number $n$, unstable modes with $c_g < 0$ exist only for counter-rotating spiral modes with $n = -1$: As shown in Fig. 5 for $A_1 = -0.5$, the most unstable mode with $n = -2$ has $c_g > 0$, as it occurs for co-rotating spiral disturbances and for axisymmetric disturbances ($n = +1$ and $n = 0$ in Fig. 5: note that the growth rate for axisymmetric disturbances is about an order of magnitude smaller than for non-axisymmetric ones).

**Figure 5:** $\gamma(\omega)$ (continuous lines) and $\alpha(\omega)$ (dashed lines) for the most unstable modes when $A_1 = -0.5$ for $x = 1$, $\Delta_0 = 0.001$ and $n = 0, +1, -2$. $N = 100$.

**REFERENCES**


