Hypersonic expansion of the Fokker–Planck equation

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A systematic study of the hypersonic limit of a heavy species diluted in a much lighter gas is made via the Fokker–Planck equation governing its velocity distribution function. In particular, two different hypersonic expansions of the Fokker–Planck equation are considered, differing from each other in the momentum equation of the heavy gas used as the basis of the expansion: in the first of them, the pressure tensor is neglected in that equation while, in the second expansion, the pressure tensor term is retained. The expansions are valid when the light gas Mach number is $O(1)$ or larger and the difference between the mean velocities of light and heavy components is small compared to the light gas thermal speed. They can be applied away from regions where the spatial gradient of the distribution function is very large, but it is not restricted with respect to the temporal derivative of the distribution function. The hydrodynamic equations corresponding to the lowest order of both expansions constitute two different *hypersonic closures* of the moment equations. For the subsequent orders in the expansions, closed sets of moment equations (hydrodynamic equations) are given. Special emphasis is made on the order of magnitude of the errors of the lowest-order hydrodynamic quantities. It is shown that if the heat flux vanishes initially, these errors are smaller than one might have expected from the ordinary scaling of the hypersonic closure. Also it is found that the *normal* solution of both expansions is a Gaussian distribution at the lowest order. One of the expansions is applied to a simplified form of the shock wave problem where an exact solution of the Fokker–Planck equations is known.

I. INTRODUCTION

Hydrodynamic descriptions for heavy species highly diluted in a gas composed of much lighter molecules [specifically, for $n_p/n \leqslant O(m/m_p)$, where the subindex p corresponds to the heavy species or small particles and m and n are the molecular mass and the number density, $m/m_p \ll 1$] are only possible in the very near-equilibrium limit $\mathrm{Kn} \ll m/m_p$. (See Ref. 1; Kn is the Knudsen number of the light gas.) In this limit, both species behave as a single fluid obeying the standard Chapman–Enskog theory for binary mixtures.[2] Otherwise, one has to solve the kinetic Boltzmann equation for the heavy component, even for near-equilibrium situations $\mathrm{Kn} \ll 1$ in which a hydrodynamic description of the light gas is still possible.[1]

The kinetic description of the heavy gas is largely simplified when its dilution is high enough ($n_p/n \ll m/m_p$) so that heavy–heavy collisions may be neglected. In this case, after a mass ratio expansion of the Boltzmann cross-collision integral, the heavy species Boltzmann equation is reduced to its Fokker–Planck (FP) form,[3]

$$\frac{\partial f'}{\partial t'} + \mathbf{u}'\cdot\nabla' f' = \tau'^{-1}\nabla_{u'}\cdot\left((\mathbf{u}'-\mathbf{W}')f' + \frac{kT'}{m_p}\nabla_{u'} f'\right), \tag{1}$$

where it has been assumed that $\mathrm{Kn} \ll 1$ and that the difference of mean velocities between both species is small compared to the speed of sound of the light gas. In the above equation f' is the heavy species velocity distribution function that is a function of time t', space coordinate $\mathbf{x}'$, and molecular velocity $\mathbf{u}'$ (unprimed variables will be subsequently used for dimensionless quantities); k is Boltzmann's constant, T' is the light gas temperature, and $\mathbf{W}'$, in this limit of high dilution of the heavy species, is related to the light gas mean velocity $\mathbf{U}'$ and to T' (both functions of t' and $\mathbf{x}'$) through

$$\mathbf{W}' = \mathbf{U}' + D\alpha_T \nabla' \ln T', \tag{2}$$

where α_T and D are the mixture thermal diffusion ratio[2] and diffusion coefficient, respectively. Here, D is related to the relaxation time τ' entering into Eq. (1) through Einstein's law:

$$\tau' = m_p D / kT'. \tag{3}$$

As a consequence of the disparity of masses, a relatively common situation is that in which the heavy gas Mach number (ratio between the velocity of the heavy gas and its own speed of sound) is much larger than 1. As we shall see, in this hypersonic limit, one can make a hypersonic expansion of Eq. (1) in such a way that a closed set of equations for the moments of f' results at each order of the expansion. Therefore, a hydrodynamic description of the dilute heavy species is still possible when $n_p/n \ll m/m_p$ (and $\mathrm{Kn} \ll 1$) in this hypersonic limit.

Defining the moments of f' as

$$n'_p \equiv \int d^3u'\, f', \tag{4a}$$

$$n'_p \mathbf{U}'_p \equiv \int d^3u'\, \mathbf{u}' f', \tag{4b}$$

$$n'_p \mathbf{U}'_p \mathbf{U}'_p + \frac{\mathbf{P}'_p}{m_p} \equiv \int d^3u'\, \mathbf{u}'\mathbf{u}' f', \tag{4c}$$

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$$m_p n'_p U'_{pi} U'_{pj} U'_{pk} + (2Q'_{pijk} + P'_{pij} U'_{pk} + P'_{pik} U'_{pj} + P'_{pjk} U'_{pi}) \equiv m_p \int d^3u'\, u'_i u'_j u'_k f', \quad (4d)$$

and so on, the equations for these moments can be written [from Eq. (1)] as

$$\frac{\partial n'_p}{\partial t} + \nabla' \cdot (n'_p \mathbf{U}'_p) = 0, \quad (5)$$

$$\frac{\partial (n'_p \mathbf{U}'_p)}{\partial t'} + \nabla' \cdot \left(n'_p U'_p U'_p + \frac{P'_p}{m_p} \right) = \frac{n'_p (\mathbf{W}' - \mathbf{U}'_p)}{\tau'}, \quad (6)$$

$$\frac{\partial P'_p}{\partial t'} + \nabla' \cdot (2Q'_p + \mathbf{U}'_p P'_p) + P'_p \cdot \nabla' \mathbf{U}'_p + (P'_p \cdot \nabla' \mathbf{U}'_p)^T = \frac{2 n'_p k}{\tau'} (T' I - T'_p), \quad (7)$$

and so forth, where I is the unit tensor and the temperature tensor is defined as $T'_p \equiv P'_p / n'_p k$ (the superscript T denotes the transposed tensor). In the hypersonic limit ($M_p \gg 1$, where M_p is the Mach number of the heavy gas), the magnitude of the mean velocity of the heavy gas U'_p is larger than its thermal speed c' by a factor of order $\mathbf{M}_p$. Therefore, the pressure tensor term in Eq. (6) and the heat flux term in Eq. (7) can be neglected relatively to the terms $n'_p \mathbf{U}'_p \mathbf{U}'_p$ and $\mathbf{U}'_p P'_p$, respectively, with errors of the order of $\mathbf{M}_p^{-1}$ for P'_p, and of order $\mathbf{M}_p^{-2}$ for the component of U'_p in the direction of the flow and for n'_p, closing the system of moment equations (hypersonic closure of the moment equations).

Hypersonic closures of the moment equations can be made at different levels of approximation. The lowest possible order is equivalent to Newton's equation of motion written in Eulerian form, and results from dropping P'_p in Eq. (6) and ignoring the higher-order moment equations. This hypersonic closure has been implicitly used in the literature of aerosol dynamics.[4] In the field of disparate-mass mixtures, it has been used to describe the impingement of seeded free jets against surfaces by Fernández de la Mora *et al.*,[5] and to describe the structure of normal shock waves (Refs. 6–8). To first order, one would neglect the heat flux term in Eq. (7) and, more generally, to order $N-1$, the $(N+1)$th moment would be dropped in the equation for the Nth moment of f'. Although not in the sense just explained, in which the hypersonic closure is only applied to the heavy species of a disparate-mass mixture and, therefore, it can be applied even for subsonic conditions of the light host gas, Hamel and Willis[9] and Edwards and Cheng[10] used the first-order hypersonic closure in the one-dimensional spherical and cylindrical expansions of a *pure* gas into a vacuum. If the Knudsen number at the source is very small, the flow becomes hypersonic before rarefaction effects become important, and these authors truncated the moment equations in this region by neglecting the heat flux term. The problem was extended to binary gas mixtures by Cooper and Bienkowski[11] and by Miller and Andres[12] who neglected the heat flux terms for *both* species.

In the next section, a systematic hypersonic expansion in the FP equation will be performed. In fact, we shall consider two expansions differing from each other in the order of the momentum equation used as the basis of the expansion: in one of them (termed *deterministic* hypersonic expansion), the lowest-order momentum equation will be Eq. (6) without the pressure tensor term, while in the second expansion (*Brownian* hypersonic expansion), the pressure tensor term will be retained in that equation at the lowest order of the expansion. Closed sets of moment equations will be obtained at each level of both expansions. We shall estimate the order of magnitude of the errors in the hydrodynamic quantities corresponding to the lowest order of these expansions and show that these errors are smaller when the heat flux vanishes initially [specifically, the error in P'_p would be $O(M_p^{-2})$ instead of $O(M_p^{-1})$], as it occurs, for instances, when the particular *normal* solution considered in Sec. III applies throughout the flow. In the last section, the deterministic hypersonic expansion will be applied to the determination of the shock wave structure in the limit of very high dilution of the heavy gas and negligible thickness of the light gas internal shock. The hypersonic results will be compared to an analytical solution of the FP equation for the shock wave problem in that limit.

II. HYPERSONIC EXPANSION OF THE FOKKER–PLANCK EQUATION. MOMENT EQUATIONS

The hypersonic expansion of Eq. (1) is based on the smallness of the heavy species thermal velocity (of the order of the speed of sound of the heavy gas) compared to the mean velocity of the heavy gas. Typically, the expansion is applied when the light gas Mach number is of order unity (or larger) and the difference between the mean velocities of both species (slip velocity) is small compared to the sound speed of the light gas [as it was assumed in the derivation of Eq. (1)[1,3]]. In these cases, the ratio among the thermal and mean speeds of the heavy gas is of order $M^{1/2}$ (or smaller), where $M \equiv m/m_p \ll 1$ is the molecular mass ratio. Therefore, one can make an expansion in the small parameter $M^{1/2}$ (see Appendix A). To that objective, Eq. (1) has to be written with the thermal velocity $\mathbf{c}'$ as the independent variable, instead of the molecular velocity $\mathbf{u}'$. However, to define the thermal velocity, we shall not use the exact value of the heavy gas mean velocity, but an approximate value $\mathbf{U}'_{p0}$ given by some level of the hypersonic closure of Eqs. (5)–(7),

$$\mathbf{c}'_0 \equiv \mathbf{u}' - \mathbf{U}'_{p0}. \quad (8)$$

Depending on how $\mathbf{U}'_{p0}$ is defined, we shall consider two different hypersonic expansions. In a first case (deterministic hypersonic expansion or DHE), $\mathbf{U}'_{p0}$ is the Newtonian deterministic velocity satisfying the equation

$$\frac{\partial \mathbf{U}'_{p0}}{\partial t'} + \mathbf{U}'_{p0} \cdot \nabla' \mathbf{U}'_{p0} = \frac{\mathbf{W}' - \mathbf{U}'_{p0}}{\tau'}, \quad (9a)$$

while in the second expansion $\mathbf{U}'_{p0}$ satisfies

$$\frac{\partial \mathbf{U}'_{p0}}{\partial t'} + \mathbf{U}'_{p0} \cdot \nabla' \mathbf{U}'_{p0} + \frac{\nabla' \cdot P'_{p0}}{m_p n'_{p0}} = \frac{\mathbf{W}' - \mathbf{U}'_{p0}}{\tau'}, \quad (9b)$$

n'_{p0} and P'_{p0} being the number density and pressure tensor resulting from Eqs. (5)–(7) with $Q'_p = 0$. In this second expansion the Brownian motion (diffusion) of the heavy molecules is taken into account at the lowest order of the expansion through the pressure tensor P'_{p0}, so that it will be called Brownian hypersonic expansion or BHE.

It would seem superfluous to consider the Brownian hypersonic expansion since, strictly speaking, the hypersonic lowest-order momentum equation is (9a). However, it has very significant advantages over the DHE regarding its range of validity (see the next section). Moreover, the lowest order of the BHE is more accurate for n'_p and $\mathbf{U}'_p$, avoiding the necessity of going to further orders in some problems (this is particularly important for the components of $\mathbf{U}'_{p0}$ that are not in the direction of the flow). On the other hand, in addition to being simpler, the main advantage of the deterministic hypersonic expansion over its Brownian counterpart is the decoupling of the lowest-order continuity and momentum equations from the pressure tensor equation. This property has been exploited in the shock wave problem to obtain an algebraic solution in phase space.[7]

Let us nondimensionalize the FP equation (1) taking into account that the heavy gas mean velocity is of the same order as the light gas thermal speed, whereas the heavy gas thermal velocity is much smaller:

$$\mathbf{x} \equiv (\mathbf{x}'/\tau_0)(m/2kT_0)^{1/2}, \quad t \equiv t'/\tau_0, \tag{10a}$$

$$f \equiv (f'/n_0)(2kT_0/m_p)^{3/2}, \quad \mathbf{c}_0 \equiv \mathbf{c}'_0 (m_p/2kT_0)^{1/2}, \tag{10b}$$

$$\mathbf{U}_{p0} \equiv \mathbf{U}'_{p0}(m/2kT_0)^{1/2}, \quad T \equiv T'/T_0, \tag{10c}$$

$$\mathbf{W} \equiv \mathbf{W}'(m/2kT_0)^{1/2}, \quad \tau \equiv \tau'/\tau_0, \tag{10d}$$

$$P_p \equiv P'_p/2n_0kT_0, \quad n_p \equiv n'_p/n_0, \tag{10e}$$

where T_0, n_0, and τ_0 are constants. Using $\mathbf{c}_0$ as an independent variable and Eqs. (9), the FP equation (1) becomes

$$\frac{\partial f}{\partial t} + \mathbf{U}_{p0}\cdot\nabla f - \left(\mathbf{c}_0\cdot\nabla\mathbf{U}_{p0} + \frac{\mathbf{c}_0}{\tau}\right)\cdot\nabla_c f - \frac{3f}{\tau} - \frac{T}{2\tau}\nabla_c^2 f = -M^{1/2}\mathbf{c}_0\cdot\nabla f, \tag{11a}$$

for the deterministic hypersonic expansion, and

$$\frac{\partial f}{\partial t} + \mathbf{U}_{p0}\cdot\nabla f - \left(\mathbf{c}_0\cdot\nabla\mathbf{U}_{p0} + \frac{\mathbf{c}_0}{\tau}\right)\cdot\nabla_c f - \frac{3f}{\tau} - \frac{T}{2\tau}\nabla_c^2 f = -M^{1/2}\left(\mathbf{c}_0\cdot\nabla f + \frac{\nabla\cdot P_{p0}}{n_{p0}}\cdot\nabla_c f\right), \tag{11b}$$

for the BHE, where ∇_c stands for the gradient in $\mathbf{c}_0$ space. From the assumptions made so far, the left-hand sides of Eqs. (11) are of order unity (or larger), while the right-hand sides are $O(M^{1/2})$, provided that the combination of spatial gradients of the distribution function appearing there are $O(1)$. Therefore, away from regions where $M^{1/2}\nabla f$ (and the corresponding right-hand side in the BHE) are $O(1)$ (or larger), we can expand f,

$$f_2 = f^{(0)} + M^{1/2} f^{(1)} + M f^{(2)} + \cdots, \tag{12}$$

in such a way that the resulting sets of moment equations at each order of the expansion become closed once the previous orders have been solved; for the unclosed nature of the system of moment equations is a consequence of the first term on the right-hand side of Eqs. (11). Notice that the expansion (12) is not limited with respect to the temporal derivatives of the distribution function f.

It is convenient to use the Fourier transform of f,

$$F(t,\mathbf{x},\mathbf{K}) \equiv \int d^3c_0 f(t,\mathbf{x},\mathbf{c}_0)\exp(-i\mathbf{K}\cdot\mathbf{c}_0). \tag{13}$$

Equations (11) then transform to

$$\frac{\partial F}{\partial t} + \mathbf{U}_{p0}\cdot\nabla F + \nabla_K F\cdot\left(\nabla\mathbf{U}_{p0}\cdot\mathbf{K} + \frac{\mathbf{K}}{\tau}\right) + F\,\nabla\cdot\mathbf{U}_{p0}\,\frac{K^2T}{2\tau}F = -iM^{1/2}\nabla\cdot\nabla_K F \tag{14a}$$

and

$$\frac{\partial F}{\partial t} + \mathbf{U}_{p0}\cdot\nabla F + \nabla_K F\cdot\left(\nabla\mathbf{U}_{p0}\cdot\mathbf{K} + \frac{\mathbf{K}}{\tau}\right) + F\,\nabla\cdot\mathbf{U}_{p0} + \frac{K^2T}{2\tau}F = -iM^{1/2}\left(\nabla\cdot\nabla_K F + \frac{\nabla\cdot P_{p0}}{n_{p0}}\cdot\mathbf{K}F\right), \tag{14b}$$

which after the expansion

$$F = F^{(0)} + M^{1/2}F^{(1)} + MF^{(2)} + \cdots, \tag{15}$$

become

$$\mathscr{L}(F^{(0)}) \equiv \frac{\partial F^{(0)}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla F^{(0)} + \nabla_K F^{(0)}\cdot\left(\nabla\mathbf{U}_{p0}\cdot\mathbf{K} + \frac{\mathbf{K}}{\tau}\right) + F^{(0)}\nabla\cdot\mathbf{U}_{p0} + \frac{K^2T}{2\tau}F^{(0)} = 0, \tag{16a}$$

at the lowest order of both expansions (however, $\mathbf{U}_{p0}$ is different in each expansion), and

$$\mathscr{L}(F^{(j)}) = -i\,\nabla\cdot\nabla_K F^{(j-1)}, \tag{16b}$$

$$\mathscr{L}(F^{(j)}) = -i\left(\nabla\cdot\nabla_K F^{(j-1)} + \frac{\nabla\cdot P_{p0}}{n_{p0}}\cdot\mathbf{K}F^{(j-1)}\right), \tag{16c}$$

for $j \geqslant 1$ and the deterministic and Brownian hypersonic expansions, respectively. We define the moments of $f^{(j)}$ as

$$n^{(j)} \equiv \int d^3c_0\, f^{(j)} = F^{(j)}(\mathbf{K}=\mathbf{0}), \tag{17a}$$

$$\mathbf{J}^{(j)} \equiv \int d^3c_0\, f^{(j)}\mathbf{c}_0 = i\,\nabla_K F^{(j)}(\mathbf{K}=\mathbf{0}), \tag{17b}$$

$$n^{(0)}\theta^{(j)} \equiv \int d^3c_0\, f^{(j)}\mathbf{c}_0\mathbf{c}_0 = -2\nabla_K\nabla_K F^{(j)}(\mathbf{K}=\mathbf{0}), \tag{17c}$$

$$q^{(j)} \equiv \int d^3c_0\, f^{(j)}\mathbf{c}_0\mathbf{c}_0\mathbf{c}_0 = -i\nabla_K\nabla_K\nabla_K F^{(j)}(\mathbf{K}=\mathbf{0}), \tag{17d}$$

and so forth. Then, the successive moment equations can easily be obtained from Eqs. (16) by just taking derivatives with respect to $\mathbf{K}$ and letting $\mathbf{K}=\mathbf{0}$. For $j = 0$, and from Eqs. (9a) and (16a), at the lowest order of the DHE we obtain

$$\frac{\partial \mathbf{U}_{p0}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla U_{p0} = \frac{\mathbf{W} - \mathbf{U}_{p0}}{\tau}, \quad (18a)$$

$$\frac{\partial n^{(0)}}{\partial t} + \nabla\cdot(n^{(0)}\mathbf{U}_{p0}) = 0, \quad (18b)$$

$$\frac{\partial \theta^{(0)}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla\theta^{(0)} + \theta^{(0)}\cdot\nabla\mathbf{U}_{p0} + (\theta^{(0)}\cdot\nabla\mathbf{U}_{p0})^T = 2\frac{TI - \theta^{(0)}}{\tau}, \quad (18c)$$

$$\frac{\partial q^{(0)}_{lmn}}{\partial t} + U_{p0i}\frac{\partial q^{(0)}_{lmn}}{\partial x_i} + q^{(0)}_{lmn}\frac{\partial U_{p0i}}{\partial x_i} + \frac{3}{\tau}q^{(0)}_{lmn} + q^{(0)}_{lmi}\frac{\partial U_{p0n}}{\partial x_i} + q^{(0)}_{lin}\frac{\partial U_{p0m}}{\partial x_i} + q^{(0)}_{imn}\frac{\partial U_{p0l}}{\partial x_i} = 0, \quad (18d)$$

and so on (repeated subindices are summed). Notice that, by definition of $\mathbf{U}_{p0}$, $\mathbf{J}^{(0)} = \mathbf{0}$. The lowest-order moment equations in the BHE are the same, except for the lowest-order momentum equation (18a) which now is [Eq. (9b)]

$$\frac{\partial \mathbf{U}_{p0}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla\mathbf{U}_{p0} + \frac{M}{2}[\nabla\cdot\theta^{(0)} + \theta^{(0)}\cdot\nabla\ln n^{(0)}] = \frac{\mathbf{W} - \mathbf{U}_{p0}}{\tau}, \quad (18e)$$

where we have identified $n_{p0} \equiv n^{(0)}$, $P_{p0} = \theta^{(0)}n^{(0)}/2$. For the subsequent orders $j \geqslant 1$, from Eq. (16b) (DHE) we have

$$\frac{\partial n^{(j)}}{\partial t} + \nabla\cdot(n^{(j)}\mathbf{U}_{p0} + \mathbf{J}^{(j-1)}) = 0, \quad (19a)$$

$$\frac{\partial \mathbf{J}^{(j)}}{\partial t} + \mathbf{J}^{(j)}\cdot\left(\nabla\mathbf{U}_{p0} + \frac{I}{\tau}\right) + \nabla\cdot(\mathbf{J}^{(j)}\mathbf{U}_{p0}) = -\frac{1}{2}\nabla\cdot(n^{(0)}\theta^{(j-1)}), \quad (19b)$$

$$\frac{\partial \theta^{(j)}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla\theta^{(j)} + \theta^{(j)}\cdot\nabla\mathbf{U}_{p0} + (\theta^{(j)}\cdot\nabla\mathbf{U}_{p0})^T - 2\frac{(n^{(j)}/n^{(0)})TI - \theta^{(j)}}{\tau} = -2\nabla\cdot q^{(j-1)}, \quad (19c)$$

$$\frac{\partial q^{(j)}_{lmn}}{\partial t} + U_{p0i}\frac{\partial q^{(j)}_{lmn}}{\partial x_i} + q^{(j)}_{lmn}\frac{\partial U_{p0i}}{\partial x_i} + \frac{3}{\tau}q^{(j)}_{lmn} + q^{(j)}_{lmi}\frac{\partial U_{p0n}}{\partial x_i} + q^{(j)}_{lin}\frac{\partial U_{p0m}}{\partial x_i} + q^{(j)}_{imn}\frac{\partial U_{p0l}}{\partial x_i} - \frac{T}{\tau}(\delta_{lm}J^{(j)}_n + \delta_{ln}J^{(j)}_m + \delta_{mn}J^{(j)}_l) = \frac{\partial\Psi^{(j-1)}_{ilnm}}{\partial x_i}, \quad (19d)$$

and so forth, where δ_{ij} is the Kronecker's delta and $\Psi^{(j)} \equiv -\nabla_K\nabla_K\nabla_K\nabla_K F^{(j)}(\mathbf{K} = \mathbf{0})$. In the BHE, the right-hand sides of the moment equations (19b)–(19d) must be replaced, respectively, by

$$\tfrac{1}{2}[n^{(j-1)}(\nabla\cdot\theta^{(0)} + \theta^{(0)}\cdot\nabla\ln n^{(0)}) - \nabla\cdot(n^{(0)}\theta^{(j-1)})], \quad (19e)$$

$$2\left(\frac{(\nabla\cdot\theta^{(0)} + \theta^{(0)}\cdot\nabla\ln n^{(0)})\mathbf{J}^{(j-1)}}{n^{(0)}} - \nabla\cdot q^{(j-1)}\right), \quad (19f)$$

$$\frac{\partial\Psi^{(j-1)}_{ilnm}}{\partial x_i} + \frac{3}{4}\theta^{(j-1)}_{nm}\frac{\partial\theta^{(0)}_{li}n^{(0)}}{\partial x_i}. \quad (19g)$$

At each order j, the system of equations (19) is closed once the previous orders have been solved.

The actual values of the moments of f [see Eqs. (4)] can be obtained from the moments defined in Eqs. (17) by realizing that $\mathbf{c} = \mathbf{c}_0 + (\mathbf{U}_{p0} - \mathbf{U}_p)$. We obtain

$$n_p = n^{(0)} + M^{1/2}n^{(1)} + Mn^{(2)} + \cdots, \quad (20a)$$

$$\mathbf{U}_p = \mathbf{U}_{p0} + (M/n_p)(J^{(1)} + M^{1/2}\mathbf{J}^{(2)} + \cdots), \quad (20b)$$

$$T_p = (n^{(0)}/n_p)(\theta^{(0)} + M^{1/2}\theta^{(1)} + M\theta^{(2)} + \cdots) - M^{-1}(\mathbf{U}_p - \mathbf{U}_{p0})(\mathbf{U}_p - \mathbf{U}_{p0}), \quad (20c)$$

$$Q_{pijk} = (1/n_p)(q^{(0)}_{ijk} + M^{1/2}q^{(1)}_{ijk} + Mq^{(2)}_{ijk} + \cdots) - M^{-3/2}(U_{pi} - U_{p0i})(U_{pj} - U_{p0j})(U_{pk} - U_{p0k}) - (1/2M^{1/2})[T_{pij}(U_{pk} - U_{p0k}) + T_{pik}(U_{pj} - U_{p0j}) + T_{pjk}(U_{pi} - U_{p0i})], \quad (20d)$$

where the heat flux tensor Q_{pijk} has been made dimensionless with $n_p kT_0(2kT_0/m_p)^{1/2}$. At zeroth order we obviously have $n_p = n^{(0)}$, $\mathbf{U}_p = \mathbf{U}_{p0}$, $T_p = \theta^{(0)}$, and $Q_p = q^{(0)}/n^{(0)}$.

Equations (18) must be solved with the initial conditions

$$n^{(0)}(t=0) = n_p(t=0),$$
$$\mathbf{U}_{p0}(t=0) = \mathbf{U}_p(t=0),$$
$$\theta^{(0)}(t=0) = T_p(t=0),$$
$$q^{(0)}(t=0) = n_p(t=0)Q_p(t=0),$$

while Eqs. (19) for $j \geqslant 1$ must be solved with the initial conditions

$$n^{(j)}(t=0) = \mathbf{J}^{(j)}(t=0) = \theta^{(j)}(t=0) = q^{(j)}(t=0) = 0.$$

According to these initial conditions, and from Eqs. (19) and (20), the order of magnitude of the errors in the lowest-order hydrodynamic quantities of the DHE are (see also Appendix A)

$$n_p = n^{(0)} + O(M), \quad (21a)$$

$$\mathbf{U}_p = \mathbf{U}_{p0} + O(M), \quad (21b)$$

$$T_p = \theta^{(0)} + O(M^{1/2}), \quad (21c)$$

since, from Eqs. (19a), $n^{(1)} = 0$ because $\mathbf{J}^{(0)} = \mathbf{0}$. [Notice that, according to the nondimensionalization (10), only the component of $\mathbf{U}_p$ in the direction of the flow is $O(1)$; the remaining components are smaller, typically $O(M^{1/2})$.] On the other hand, for the BHE, we have

$$n_p = n^{(0)} + O(M^{3/2}), \quad (22a)$$

$$\mathbf{U}_p = \mathbf{U}_{p0} + O(M^{3/2}), \quad (22b)$$

$$T_p = \theta^{(0)} + O(M^{1/2}), \quad (22c)$$

since from Eq. (19b) with the right-hand side given by Eq. (19e), $\mathbf{J}^{(1)} = \mathbf{0}$, so that $n^{(2)} = 0$. If the heat flux tensor vanishes initially, from Eq. (18d) $q^{(0)} = 0$; whence, from Eq. (19c) $\theta^{(1)} = 0$ (notice that $n^{(1)} = 0$), so that the error at the lowest order in T_p is substantially reduced to $O(M)$. In addition, $\mathbf{J}^{(2)} = \mathbf{0}$ from Eq. (19b) and $n^{(3)} = 0$ from Eq. (19a). (These results are valid for both the DHE and the BHE.) Thus, in the BHE, $n_p = n^{(0)} + O(M^2)$ and $\mathbf{U}_p$

$= \mathbf{U}_{p0} + O(M^2)$. In the DHE, the errors for n_p and $\mathbf{U}_p$ at the lowest-order remain unchanged because $\mathbf{J}^{(1)} \neq \mathbf{0}$ and $n^{(2)} \neq 0$.

III. NORMAL SOLUTION

In the preceding section we expanded the FP equations (11) in powers of $M^{1/2}$. However, instead of solving the resulting equations (16) for the Fourier transform of the distribution function, we were contented with obtaining the moment equations for the successive orders in the expansion. In the present section, an important particular solution to Eqs. (11) (termed the normal solution) is given, whose lowest order is an anisotropic Gaussian distribution. Wherever this normal solution applies, the error in the lowest-order temperature tensor is smaller by a factor $M^{1/2}$ than that given by Eqs. (21) and (22), since $\boldsymbol{q}^{(0)} = \boldsymbol{0}$.

In fact, the function

$$F_{\mathrm{G}}^{(0)} \equiv n^{(0)} \exp(-\mathbf{KK}{:}\theta^{(0)}/4), \qquad (23)$$

with $n^{(0)}(\mathbf{x},t)$ and $\theta^{(0)}(\mathbf{x},t)$ governed by Eqs. (18b) and (18c), is a particular solution to Eq. (16a). Since the inverse Fourier transform of (23) is

$$f_{\mathrm{G}}^{(0)} \equiv [n^{(0)}/\pi^{3/2}(\det \theta^{(0)})^{1/2}] \exp(-\mathbf{c}_0\mathbf{c}_0{:}\theta^{(0)-1}), \qquad (24)$$

at the lowest order we have a Gaussian distribution with number density $n^{(0)}$, mean velocity $\mathbf{U}_{p0}$, and temperature tensor $\theta^{(0)}$ satisfying the hydrodynamic equations corresponding to a hypersonic closure of the moment equations. Notice that the solution (24) is valid for both the DHE and the BHE; the difference resides in the lowest-order equation for $\mathbf{U}_{p0}$.

It is shown in Appendix B that, if at $t = 0$ [or at $\mathbf{x} = \mathbf{x}(0)$ for stationary problems] $F^{(0)}$ is of the form given by Eq. (23), the general solution to Eq. (16a) is everywhere given by Eqs. (23) and (18b) and (18c). If, in addition, one were able to show that any solution of Eq. (16a) would tend to Eq. (23) as $t \to \infty$, $F_{\mathrm{G}}^{(0)}$ could properly be called the normal solution of Eq. (16a). However, we have not succeeded yet in obtaining the general solution of Eq. (16a), except for some particular cases (see Appendix B), most of them satisfying the condition $F^{(0)} \to F_{\mathrm{G}}^{(0)}$ as $t \to \infty$. In any case, we shall term $F_{\mathrm{G}}^{(0)}$, and the solution at the subsequent orders derived from it, the normal solution of the hypersonic expansion.

Let us consider the BHE. Making use of Eq. (23), the first-order equation [Eq. (16c) for $j = 1$] may be written as

$$\mathscr{L}\{F^{(1)}\} = -(i/8)\mathbf{K}\cdot\theta^{(0)}\cdot\nabla\theta^{(0)}{:}\mathbf{KK}F_{\mathrm{G}}^{(0)}. \qquad (25)$$

The general solution of this equation is the sum of the general solution of the homogeneous equation (16a) (which, therefore, can be included into the lowest-order solution), plus a particular solution cubic in $\mathbf{K}$:

$$F^{(1)} = (iq_{lmn}^{(1)} K_l K_m K_n / n^{(0)}) F_{\mathrm{G}}^{(0)}, \qquad (26)$$

where the proportionality *constant* $\boldsymbol{q}^{(1)}(t,\mathbf{x})$ obviously coincides with $\boldsymbol{q}^{(1)}$ as defined by (17d) and satisfies Eq. (19d) with the right-hand side substituted by Eq. (19g).

Similarly, the particular solution $F^{(j)}$ for the subsequent orders $j > 1$ is polynomical in $\mathbf{K}$, whose constants are the moments of f entering at that level of the expansion satisfying the moment equations given in the preceding section. Therefore, there is not much gain in pursuing this procedure beyond $j = 1$. Nevertheless, the above normal solutions for $j = 0$ and $j = 1$ yield additional information not completely contained in the moment method of the preceding section. First of all, consider the order of magnitude of the errors in the lowest-order hydrodynamic quantities. Since $\boldsymbol{q}^{(0)} = \boldsymbol{0}$, we have $\theta^{(1)} = \boldsymbol{0}$, $\mathbf{J}^{(2)} = \mathbf{0}$, and $n^{(3)} = 0$, and from Eqs. (20) we have (BHE)

$$n_p = n^{(0)} + O(M^2), \qquad (27a)$$

$$\mathbf{U}_p = \mathbf{U}_{p0} + O(M^2), \qquad (27b)$$

$$T_p = \theta^{(0)} + O(M). \qquad (27c)$$

(As was discussed in the preceding section, this reduction in the errors of the lowest-order hydrodynamic quantities in the BHE is just a consequence of $\boldsymbol{q}^{(0)} = \boldsymbol{0}$, which is satisfied if the heat flux vanishes initially, independent of whether the normal solution is valid.) On the other hand, when the normal solution applies, the procedure of solving the hierarchy of moment equations (19) is enormously simplified because all the highest-order moments are known functions of the hydrodynamic quantities $n^{(0)}$ and $\theta^{(0)}$ (for instance, $\Psi^{(0)} = -\frac{3}{4}n^{(0)}\theta^{(0)}\theta^{(0)}$). Finally, since the gradient of $n^{(0)}$ does not appear in the right-hand side of Eq. (25), the Brownian hypersonic expansion with $F^{(0)} = F_{\mathrm{G}}^{(0)}$ fails inside density boundary layers in which $|\nabla n_p| = O(M^{-1})$, instead of $O(M^{-1/2})$ as one might have expected from considering Eq. (14b) alone, thus broadening the range of applicability of the expansion. In real velocity space, the right-hand side of Eq. (25) reads [from Eq. (11b)]:

$$f_{\mathrm{G}}^{(0)}\mathbf{c}_0\cdot[\tfrac{1}{2}\nabla \ln(\det \theta^{(0)}) + \nabla\theta^{(0)-1}{:}\mathbf{c}_0\mathbf{c}_0 + \theta^{(0)-1}\cdot(\nabla\cdot\theta^{(0)})]. \qquad (28)$$

The simplification in the expansion due to the normal solution also applies, obviously, to the DHE. However, the extension in the validity range of the hypersonic expansion with respect to the density gradients is a consequence of the inclusion of the pressure tensor term in the lowest-order momentum equation and thus it does not apply to the DHE. The reduction of the errors at the lowest order of n_p and $\mathbf{U}_p$ is also a consequence of this inclusion, so that, in the DHE, $n_p = n^{(0)} + O(M)$ and $\mathbf{U}_p = \mathbf{U}_{p0} + O(M)$ (see the end of the preceding section), even if the normal solution is valid. Nevertheless, $T_p = \theta^{(0)} + O(M)$, as in the BHE.

IV. APPLICATION TO THE SHOCK WAVE PROBLEM

In this section, the DHE is applied to the determination of the structure of a normal shock wave in a heavy gas diluted in a much lighter gas.[6–8] In particular, we consider the limit in which the internal shock of the light gas has negligible width (it is a discontinuity occurring at $x = 0$) and the heavy gas is so diluted that the light gas properties remain constant for $x > 0$. We apply the hypersonic expansion to this oversimplified case because there is an analytical solution of the FP equation[13] with which the hypersonic results may be compared. On the other hand, we use the DHE because the hydrodynamic equations corresponding to the lowest order of this expansion were used in Refs. 6–8, yielding an algebraic phase space solution for the velocity of the heavy gas (see Ref. 7). Here we also give the first-order hypersonic correction. Notice that the normal solution given in the last section applies throughout the shock, since upstream ($x \to -\infty$) the mixture is in equilibrium, and therefore the distribution function is Maxwellian (isotropic Gaussian).

For this shock wave problem, the hydrodynamic equations corresponding to the lowest and first orders of the deterministic hypersonic expansion can be written as [Eqs. (18) and (19); the equations are, obviously, for $x > 0$]

$$n^{(0)}U_{p0} = \text{const} \equiv U_p(0), \tag{29a}$$

$$\frac{dU_{p0}}{dx} = \frac{W}{U_{p0}} - 1, \tag{29b}$$

$$\frac{d\theta_{\parallel}^{(0)}}{dx} = 2\,\frac{T - \theta_{\parallel}^{(0)}(W/U_{p0})}{U_{p0}}, \tag{29c}$$

$$\frac{d\theta_{\perp}^{(0)}}{dx} = 2\,\frac{T - \theta_{\perp}^{(0)}}{U_{p0}}, \tag{29d}$$

$$q_{\parallel}^{(0)} = q_{\perp}^{(0)} = 0, \tag{29e}$$

$$n^{(1)} = \theta_{\parallel}^{(1)} = \theta_{\perp}^{(1)} = 0, \tag{30a}$$

$$\frac{dJ^{(1)}}{dx} = -\frac{J^{(1)}}{U_{p0}}\left(\frac{2W}{U_{p0}} - 1\right) - \frac{n^{(0)}}{2U_{p0}}\left(\frac{d\theta_{\parallel}^{(0)}}{dx} - \frac{\theta_{\parallel}^{(0)}}{U_{p0}}\frac{dU_{p0}}{dx}\right), \tag{30b}$$

$$\frac{dq_{\parallel}^{(1)}}{dx} = -\frac{q_{\parallel}^{(1)}}{U_{p0}}\left(\frac{4W}{U_{p0}} - 1\right) + \frac{3J^{(1)}}{U_{p0}} - \frac{3}{4U_{p0}}\frac{d}{dx}\left(n^{(0)}\theta_{\parallel}^{(0)}\theta_{\parallel}^{(0)}\right), \tag{30c}$$

$$\frac{dq_{\perp}^{(1)}}{dx} = -\frac{q_{\perp}^{(1)}}{U_{p0}}\left(\frac{2W}{U_{p0}} + 1\right) + \frac{J^{(1)}}{U_{p0}} - \frac{1}{4U_{p0}}\frac{d}{dx}\left(n^{(0)}\theta_{\parallel}^{(0)}\theta_{\perp}^{(0)}\right), \tag{30d}$$

where $\theta_{\parallel} \equiv \theta_{xx}$, $\theta_{\perp} \equiv \theta_{yy} = \theta_{zz}$, $q_{\parallel} \equiv q_{xxx}$, $q_{\perp} \equiv q_{xyy} = q_{xyy}$. To complete the first-order correction to the lowest-order approximation, we also need the equations for $n^{(2)}$ and $\theta^{(2)}$:

$$n^{(2)} = -J^{(1)}/U_{p0}, \tag{31a}$$

$$\frac{d\theta_{\parallel}^{(2)}}{dx} = -2\,\frac{\theta_{\parallel}^{(2)}W}{U_{p0}^2} + \frac{2n^{(2)}}{n^{(0)}U_{p0}} - \frac{2}{U_{p0}}\frac{dq_{\parallel}^{(1)}}{dx}, \tag{31b}$$

$$\frac{d\theta_{\perp}^{(0)}}{dx} = -2\,\frac{\theta_{\perp}^{(2)}}{U_{p0}^2} + \frac{2n^{(2)}}{n^{(0)}U_{p0}} - \frac{2}{U_{p0}}\frac{dq_{\perp}^{(1)}}{dx}. \tag{31c}$$

Notice that we have used T'^{-}, $n_p'^{-}$, and τ'^{+} as the constants T_0, n_0, and τ_0, respectively, in Eqs. (10), where the superscripts $-$ and $+$ stand for the conditions upstream and downstream of the shock (τ' is now constant since the light gas properties are constants).

Equations (29)–(31) must be solved with the following boundary conditions at $x = 0$:

$$U_{p0} = U_p(0), \quad \theta_{\parallel}^{(0)} = \theta_{\perp}^{(0)} = 0, \tag{32a}$$

$$J^{(1)} = q_{\parallel}^{(1)} = q_{\perp}^{(1)} = 0, \tag{32b}$$

$$n^{(2)} = \theta_{\parallel}^{(2)} = \theta_{\perp}^{(2)} = 0. \tag{32c}$$

In terms of the upstream Mach number of the light gas,

$$\text{Ma} \equiv U'^{-}/(\gamma k T'^{-}/m)^{1/2}, \tag{33}$$

where γ is the specific heat ratio of the light gas, we have $U_p(0) = (\gamma/2)^{1/2}\,\text{Ma}$. On the other hand, W and T can be related to Ma through the Rankine–Hugoniot conditions for the internal shock of the light gas (discontinuity at $x = 0$):

$$W \equiv \frac{U'^{+}}{(2kT'^{-}/m)^{1/2}} = \left(\frac{\gamma}{2}\right)^{1/2}\text{Ma}\,\frac{[\text{Ma}^2(\gamma - 1) + 2]}{\text{Ma}^2(\gamma + 2)}, \tag{34a}$$

$$T \equiv \frac{T'^{+}}{T'^{-}} = 1 + \frac{2(\gamma - 1)(\text{Ma}^2 - 1)(\gamma\,\text{Ma}^2 + 1)}{(\gamma + 1)^2\,\text{Ma}^2}. \tag{34b}$$

Using Eqs. (20), up to first order in M, we have

$$n_p = n^{(0)} + Mn^{(2)}, \tag{35a}$$

$$U_p = U_{p0} + (M/n^{(0)})J^{(1)}, \tag{35b}$$

$$T_{p\parallel} = \frac{n^{(0)}}{n^{(0)} + Mn^{(2)}}\,\theta_{\parallel}^{(0)} + M\left[\theta_{\parallel}^{(2)} - \left(\frac{J^{(1)}}{n^{(0)}}\right)^2\right], \tag{35c}$$

$$T_{p\perp} = [n^{(0)}/(n^{(0)} + Mn^{(2)})]\,\theta_{\perp}^{(0)} + M\theta_{\perp}^{(2)}, \tag{35d}$$

$$Q_{p\parallel} = \frac{M}{n^{(0)}U_{p0}}\left(\frac{q_{\parallel}^{(1)}}{\theta_{\parallel}^{(0)}} - \frac{3J^{(1)}}{2}\right), \tag{35e}$$

$$Q_{p\perp} = \frac{M}{n^{(0)}U_{p0}}\left(\frac{q_{\perp}^{(1)}}{\theta_{\perp}^{(0)}} - \frac{J^{(1)}}{2}\right), \tag{35f}$$

where $Q_{p\parallel} \equiv Q'_{pxxx}/U'_p\,P'_{pxx}$ and $Q_{p\perp} \equiv Q'_{pxyy}/U'_p\,P'_{pyy} = Q'_{pxzz}/U'_p\,P'_{pzz}$. We have plotted in Figs. 1(a)–1(e) the quantities (35) at the lowest order and first order in M for a He–Xe mixture ($M = 0.0304$) and a He–Ar mixture ($M = 0.1$) with Ma $= 1.5$. We have also included the results from Ref. 13 where the FP equation for this problem is solved analytically by means of an eigenexpansion whose coefficients are obtained with errors of order $\exp(-1/M)$ [$O(10^{-6})$ for He–Ar and $O(10^{-15})$ for He–Xe]. As predicted, the difference between the results from the lowest order of the deterministic hypersonic expansion and the exact FP results for n_p, U_p, and T_p remain $O(M)$ or smaller. The results at the first order in M practically coincide with the FP results, except for $T_{p\parallel}$ just after its maximum for the He–Xe mixture [Fig. 1(b)].

The reason for the excellent agreement between the hypersonic approximation and the FP results even downstream of the shock where the Mach number of the heavy gas is not very large (for a He–Ar mixture with Ma $= 1.5$, the downstream value of the heavy gas Mach number is 3.38, while for a He–Xe mixture it is 6.13), is the approach to equilibrium of the mixture as $y \to \infty$, so that the Gaussian distribution corresponding to the lowest order of the hypersonic expansion becomes exact as $y \to \infty$. Hence the lowest order of the hypersonic expansion works very well throughout the shock. The subsequent corrections are more accurate

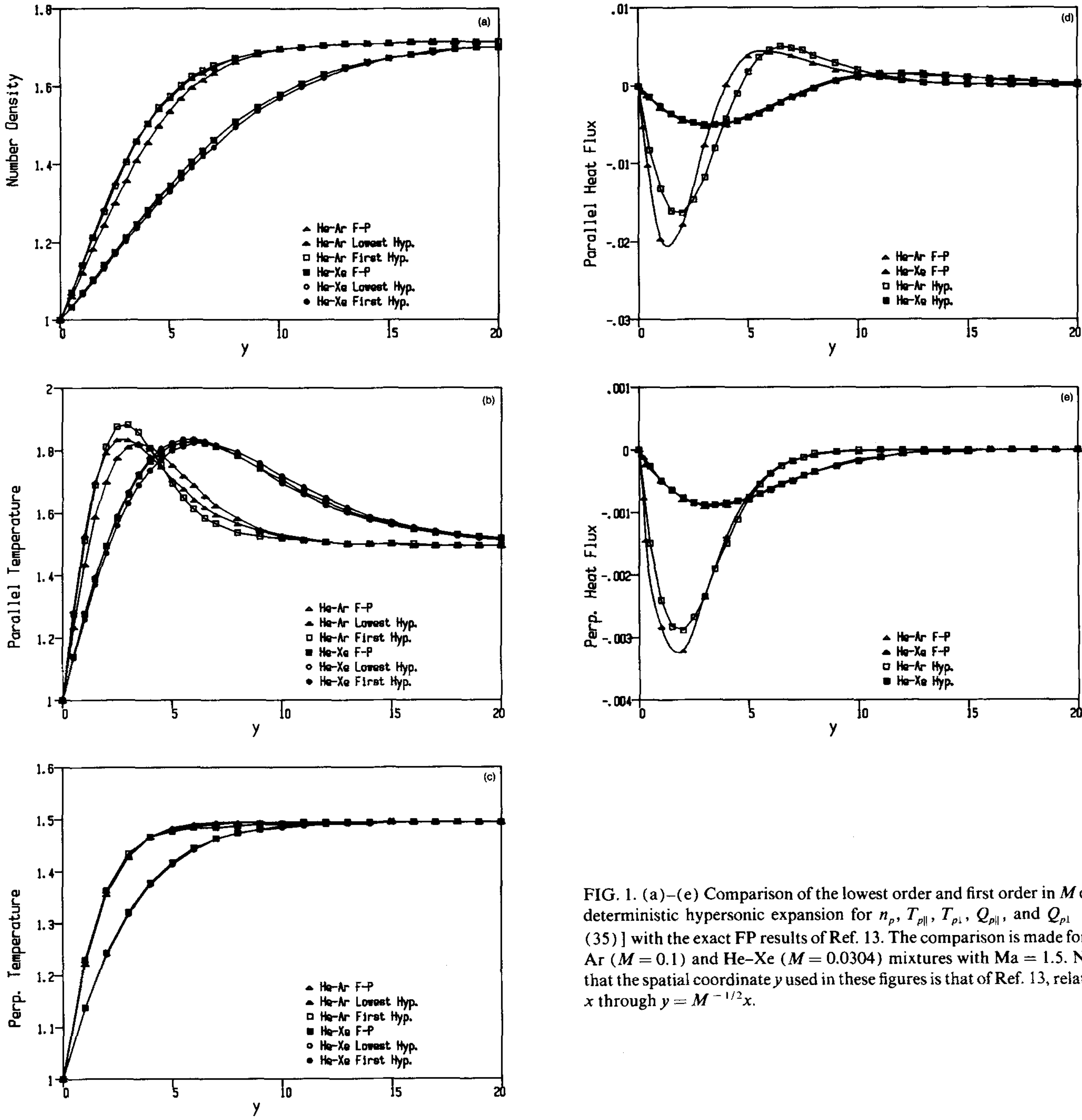

FIG. 1. (a)–(e) Comparison of the lowest order and first order in M of the deterministic hypersonic expansion for n_p, $T_{p\parallel}$, $T_{p\perp}$, $Q_{p\parallel}$, and $Q_{p\perp}$ [Eqs. (35)] with the exact FP results of Ref. 13. The comparison is made for He–Ar ($M = 0.1$) and He–Xe ($M = 0.0304$) mixtures with Ma = 1.5. Notice that the spatial coordinate y used in these figures is that of Ref. 13, related to x through $y = M^{-1/2}x$.

in the head than in the tail of the shock [as it is observed in Figs. 1(a)–1(c)]. A direct comparison of the Gaussian distribution (24) corresponding to the lowest order of the hypersonic expansion with the distribution function obtained by the direct solution of the FP equation can be found in Ref. 13.

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APPENDIX A: ERRORS AT THE LOWEST ORDER AND VALIDITY OF THE HYPERSONIC TRUNCATION

In Secs. II and III it has been assumed, without any loss in generality, that U_p is of the same order as U, and that the Mach number of the light gas is $O(1)$, so that $M_p = O(M^{-1/2})$. The reason for this has been the use of the mass ratio (more particularly, $M^{1/2}$) as the small parameter of the expansion, instead of M_p^{-1}. Of course, all the results remain the same if one substitutes $M^{1/2}$ by M_p^{-1}. This substitution may be particularly convenient when expressing the errors at the lowest order of the expansion [Eqs. (21), (22), and (27)]. In addition, one must also take into account the gradient of the distribution function appearing on the right-hand side of Eq. (11) (which has been assumed of order unity) when expressing these errors. With the dimensionless variables defined in (10), the gradient terms multiplying $M^{1/2}$ in Eqs. (11) are of order

$$S \equiv [\tau_0(2kT_0/m)^{1/2}]/L, \tag{A1}$$

where L is a length characterizing the spatial variations of n_p and T_p:

$$L = \min(|\nabla' \ln T'_p|^{-1},\ |\nabla' \ln n'_p|^{-1}). \tag{A2}$$

(Notice that, according to Sec. III, the gradients of the density are not important for the BHE when the normal solution applies.) The orders of magnitude given in Secs. II and III must then be multiplied by the Stokes number (A1), making it explicit that the hypersonic expansion ceases to be valid in regions where the spatial gradients are so large that $SM_p^{-1} = O(1)$ or, in the case when the heat flux vanishes initially, $SM_p^{-2} = O(1)$.

APPENDIX B: SOLUTION OF EQ. (16a)

If we write $F^{(0)}(t,\mathbf{x},\mathbf{K}) = \psi(t,\mathbf{x},\mathbf{K})F_G^{(0)}$, where $F_G^{(0)}$ is given by Eq. (23), the characteristic equations associated to the lowest-order Eqs. (16a) and (18a) are (DHE)

$$\frac{d\mathbf{x}}{dt} = \mathbf{U}_{p0}, \tag{B1}$$

$$\frac{d\mathbf{K}}{dt} = (\nabla \mathbf{U}_{p0})\cdot\mathbf{K} + \frac{\mathbf{K}}{\tau}, \tag{B2}$$

$$\frac{d\mathbf{U}_{p0}}{dt} = \frac{(\mathbf{W} - \mathbf{U}_{p0})}{\tau}, \tag{B3}$$

$$\frac{d(\nabla \mathbf{U}_{p0})}{dt} = \frac{\nabla\mathbf{W} - \nabla\mathbf{U}_{p0}}{\tau} - (\nabla\mathbf{U}_{p0})\cdot(\nabla\mathbf{U}_{p0}), \tag{B4}$$

$$\frac{d\psi}{dt} = 0. \tag{B5}$$

Hence the general solution of Eq. (16a) may be written as

$$F^{(0)}(\mathbf{x},\mathbf{K},t) = \psi[\xi_1(\mathbf{x},\mathbf{K},t), \xi_2(\mathbf{x},\mathbf{K},t), \xi_3(\mathbf{x},\mathbf{K},t)]\, F_G^{(0)}(\mathbf{x},\mathbf{K},t), \tag{B6}$$

where the ξ_i are the invariants of Eq. (B2) [the invariants of Eq. (B1) depend only on $\mathbf{x}$ and t and can be included in $F_G^{(0)}$]. If [$F^{(0)}$ is Gaussian at $t=0$ [or $\mathbf{x} = \mathbf{x}(0)$ for stationary problems], ψ is a constant and, from Eq. (B6), $F^{(0)}$ will be described by Eqs. (23), (18b), and (18c) throughout the flow. Therefore, the lowest-order distribution $f^{(0)}$ will always be a Gaussian distribution if it is Gaussian at $t=0$.

The solution (B6) (and, therefore, all the results given in this Appendix) is also valid for the BHE since Eqs. (B1), (B2), and (B5) remain the same [we have the same Eq. (16a) at the lowest order]. However, because the inclusion of the pressure tensor in the momentum equation (B3) (which, thus, is no longer *deterministic*), it is not possible to obtain a closed set of ordinary differential equations like Eqs. (B1)–(B5) at the lowest order of the BHE.

We have not succeeded in obtaining the general form of the invariants of Eq. (B2) but for two particular cases. Let us write Eq. (B2) as

$$\frac{\partial\mathbf{K}}{\partial t} + \mathbf{U}_{p0}\cdot\nabla\mathbf{K} = (\nabla\mathbf{U}_{p0})\cdot\mathbf{K} + \frac{\mathbf{K}}{\tau}. \tag{B7}$$

For a spatially homogeneous problem, $\mathbf{U}_{p0}$ is a constant and the invariants are

$$\xi_i = K_i \exp\left(-\int_0^t \frac{dt}{\tau}\right). \tag{B8}$$

Hence, since $\tau > 0$, the solution (B6) decays to $F_G^{(0)}$ as $t \to \infty$. The other particular case in which we find a solution of (B7) is the stationary one-dimensional problem in which

$$U_{p0}\frac{dK}{dx} = K\frac{dU_{p0}}{dx} + \frac{K}{\tau},$$

so that

$$K = \xi U_{p0}(x)\exp\left(\int_0^x \frac{dx}{U_{p0}(x)\tau(x)}\right)$$

and the invariant is (making the use of $dx = U_{p0}\, dt$)

$$\xi = \frac{K}{U_{p0}(t)}\exp\left(-\int_0^t \frac{dt}{\tau(t)}\right).$$

For this case, it is not clear whether the general solution of Eq. (16a) tends exponentially to the function (23) as t goes to infinity, because there is a t dependence in $U_{p0}(t)$. For example, for linear flows of the light carrier gas, $W = U = ax$, with $\tau = \tau_0 =$ const, it can be shown that ξ decays to zero as $t \to \infty$ only if $a > -\frac{1}{4}$.[14] Otherwise ξ oscillates, but never diverges, as $t \to \infty$.

Furthermore, in general, if $\nabla\mathbf{U}_{p0}$ is a symmetric tensor,

$$\mathbf{K} = \xi_1\mathbf{U}_{p0}\exp\left(\int_0^t \frac{dt}{\tau}\right)$$

is a solution of (B7) for the stationary case (notice that $\mathbf{U}_{p0}\cdot\nabla = d/dt$), though the other two invariants are not so easily obtained. The symmetry of $\nabla\mathbf{U}_{p0}$ is guaranteed automatically for problems where $\mathbf{U}_{p0}$ is irrotational upstream [$\mathbf{x} = \mathbf{x}(0)$] and $\mathbf{W}$ is potential through the flow field, as can be seen by taking the curl of Eq. (18a). Again, that ξ_1 goes to zero as $t \to \infty$ depends on the form of $\mathbf{U}_{p0}(t)$.

We conclude that, at least for some situations, the lowest-order distribution $f^{(0)}$ decays exponentially to a Gaussian distribution, so that Eq. (24) is a very good approximation for $t' > \tau'_0$ (τ'_0 is the initial value of τ') regardless of the initial distribution. On the other hand, Eq. (24) is valid for all $t \geqslant 0$ if at $t=0$ the distribution is Gaussian, as it occurs, for instance, in the shock problem of Sec. IV.

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