Nonlocal electron heat flux revisited

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A known nonlocal model of electron heat flux, applying for (scale length/thermal ion–electron mean-free path) of order $Z_i^{1/2}(\epsilon^*/T)^{3/2}$, ionization number Z_i large, and $\epsilon^* \sim 6.5$ T (the energy of electrons carrying most of the flux), is reconsidered. The large ϵ^*/T ratio simplifies the complete formalism. A simple flux formula, exact for both smooth and steep profiles, is given. Thermoelectric effects and other models are discussed.

Heat transport is essential to the physics of laser targets. Albritton *et al.*[1] gave a self-consistent calculation of nonlocal heat flux q in a weakly collisional regime. In the present work we reconsider their model and give new results and simplifications.

We first review the model conditions not quite stated in Ref. 1. The kinetic equation for the electron distribution function f (inhomogeneous along x) reads $v_x\, \partial f/\partial x - (eE/m_e)\partial f/\partial v_x = C_{ei} + C_{ee}$. We use the ion frame, so $C_{ei} = v\lambda_{ei}^{-1}(\partial/\partial\mu)(1-\mu^2)\partial f/\partial\mu$ plus terms of order m_e/m_i;[2] here $\mu \equiv v_x/v$ and $\lambda_{ei} \equiv m_e^2 v^4/2\pi Z_i n e^4 \ln \Lambda_{ei}$, the symbols being as usual. In the free-streaming operator, and based on both the small value of the mass ratio and flow quasineutrality, (i) we have neglected the time derivative $\partial/\partial t$ and terms arising in the ion frame from ion hydrodynamics. The energy dependence of mean-free paths allows us to assume that (ii) main-body electron are near-Maxwellian, but (iii) those contributing dominantly to q, characterized by an energy $\epsilon^* \sim 6.5$ T,[3] are not. Taking Z_i large allows us to also assume (iv) that ϵ^* electrons are nonetheless near-isotropic. For $\epsilon \sim \epsilon^*$ and $f \simeq f_0$ (isotropic),[2] $C_{ee} = 2v^2\lambda_{ee}^{-1}(\partial/\partial v)\ [f_0 + (T/m_e v)\partial f_0/\partial v]$ with $\lambda_{ee} \ln \Lambda_{ee} \equiv Z_i \lambda_{ei} \ln \Lambda_{ei}$.

Assumption (iii) determines the regime of interest

$$H \sim (\lambda_{ei}^* \lambda_{ee}^*)^{1/2} \sim Z_i^{1/2} (\epsilon^*/T)^2 \lambda_{ei}^T, \quad (\epsilon^*/T)^2 \sim 40, \tag{1}$$

with $H \equiv$ scale length and λ^*, λ^T being λ at energies ϵ^* and T, respectively; (ii) and (iv) require $\lambda_{ei}^T \lambda_{ee}^T/H^2$ and $(\lambda_{ei}^*/H)^2$ to be small, that is, $(\epsilon^*/T)^4 \gg 1$ and $Z_i \gg 1$. Neglect of the free-streaming term (i) requires $(m_i/Z_i m_e)^{1/2} \gg (Z_i T/\epsilon^*)^{1/2}$. Using x and $\epsilon \equiv \frac{1}{2} m_e v^2 - e\phi(x)$ as independent variables, the kinetic equation and its angle average yield both $f - f_0 = -\mu \frac{1}{2}\lambda_{ei}\, \partial f_0/\partial x$ and an equation for f_0 itself, which is simplified by using $\epsilon^* \gg T$: The ansatz $\partial \ln|f_0 - f_M|/\partial\epsilon \ll T^{-1}$, where $f_M \equiv n(m_e/2\pi T)^{3/2} \times \exp[-(\epsilon + e\phi)/T]$, allows us to drop the last term in $C_{ee} \alpha (\partial/\partial\epsilon)[f_0 - f_M + T\partial(f_0 - f_M)/\partial\epsilon]$; the ansatz is satisfied, for example, by a power law but not by f_M. Also, since electron-momentum balance gives $e\phi \sim T$, one writes $\epsilon + e\phi \simeq \epsilon$ for $\frac{1}{2} m_e v^2$ when appearing in powers within the f_0 equation. For a scale length larger than (1) one obtains $f_0 = f_M$, exactly recovering classical results, E-field effects included. In general,

$$\frac{\partial f_0}{\partial \epsilon} - \frac{\partial f_M}{\partial \epsilon} = -\epsilon^3 \frac{\partial^2 f_0}{\partial \xi^2}, \tag{2}$$

where $d\xi \equiv (6 Z_i \ln \Lambda_{ei} \ln \Lambda_{ee})^{1/2} \pi e^4 n\, dx$ and $Z_i \simeq$ const. For an infinite plasma the immediate solution to this "heat-diffusion" equation, with $-\epsilon^4/4$ as a time-like variable and f_0 vanishing as $\epsilon \to \infty$, was given in Ref. 1,

$$f_0(\xi,\epsilon) = \int \frac{d\xi'}{\pi^{1/2}} \int_\epsilon^\infty \frac{f_M(\xi',\epsilon')d\epsilon'}{T'(\epsilon'^4 - \epsilon^4)^{1/2}} \exp\left(\frac{-(\xi-\xi')^2}{\epsilon'^4 - \epsilon^4}\right), \tag{3}$$

a solution that satisfies the ansatz; here $T' \equiv T(\xi')$.

Generally, $\epsilon^*(T)/T'$ will be large. Clearly, this is so if the high and low temperatures (T_h, T_l) of the profile are comparable, or if $T_l \ll T_h$ and $T \sim T_h$; for the final case, $T \sim T_l \ll T_h$, note that if (1) is satisfied at the top of the profile, $\epsilon^*(T_h)$ electrons will usually be collision dominated at the high densities, $n \sim T_h n(T_h)/T_l$, prevailing at the bottom. Let us now make the ansatz $\xi_H^2 T' \ll [\epsilon^*(T)]^5$, with $\xi_H/H \equiv d\xi/dx$ evaluated at the density $n(T)$ and rewrite (3),

$$f_0(\xi,\epsilon) = \int \frac{f_M(\xi',\epsilon)d\xi'}{4\pi^{1/2}\epsilon T'}\, Q\left(\frac{\epsilon}{T'}, \frac{T'(\xi-\xi')^2}{\epsilon^5}\right),$$

$$Q(a,b) \equiv \int_1^\infty \frac{\exp[-a(y^{1/4}-1) - ab/(y-1)]}{y^{3/4}(y-1)^{1/2}}\, dy.$$

Here a, b^{-1} are large, so only values ϵ' close to ϵ $(y \simeq 1)$ contribute to f_0; we have

$$Q \simeq \int_1^\infty (y-1)^{-1/2}\, dy \exp\left(-\frac{1}{4}a(y-1) - \frac{ab}{(y-1)}\right)$$

$$= 2\left(\frac{\pi}{a}\right)^{1/2} \exp(-ab^{1/2}),$$

leading to

$$f_0(\xi,\epsilon) = \int \frac{f_M(\xi',\epsilon)d\xi'}{2(\epsilon^3 T')^{1/2}} \exp\left(\frac{-|\xi-\xi'|}{(\epsilon^3 T')^{1/2}}\right). \tag{4}$$

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The parameter range of interest is clearly $\xi_H^2 \sim [\epsilon^*(T)]^3 T'$, which is equivalent to

$$H \sim Z_i^{1/2} (\epsilon^*/T)^{3/2} \lambda_{ei}^T, \quad (\epsilon^*/T)^{3/2} \sim 15, \qquad (1')$$

a slight correction to (1), arising from the fact that the effective energy width in Eq. (2) is T not ϵ^*. We recall that the overall balance of momentum and energy in the corona blowing off a laser target yields a typical distance from ablation to critical surface $H \sim (m_i/Z_i m_e)^{1/2} \lambda_{ei}^T$,[4] which numerically agrees with (1'). For that range our ansatz follows from $T' \ll \epsilon^*(T)$; also, Eq. (4) at thermal energies, $\epsilon \sim T$, is a convolution of $f_M(\xi', \epsilon)$ and a δ function, giving $f_0 \simeq f_M$. For smoother profiles (4) gives $f_0 \simeq f_M$ at $\epsilon \sim \epsilon^*(T)$, as it should.

Equation (3) and standard formulas were used in Ref. 1 to obtain electron particles and heat fluxes

$$\left\{ nu, q + \frac{5}{2} nuT \right\} = \frac{-(\lambda_{ei}/\lambda_{ee})^{1/2}}{4\pi(3m_e)^{1/2}} \int \frac{dx'\, n'}{T'^{1/2}} \{1, T'\} \times \left(\{I^*, K^*\} \frac{dT'}{dx'} + \{J^*, L^*\} eE'_{nl} \right), \qquad (5)$$

where $eE_{nl} \equiv eE + Td \ln n/dx - \frac{5}{2}\, dT/dx$. Albritton *et al.*[1] found kernels I, J, K, and L given as four double integrals,

$$\theta^{2+2\beta} \int_0^\infty dy\, y^\beta \exp(-\theta^{1/2} y^{1/4}) \times \int_0^1 dy'\, y'^\alpha (1-y')^{1/2} \exp\left(\frac{-1}{y(1-y')} \right)$$

for different α, β; here $\theta \equiv |\xi - \xi'|/T'^2$. The term $\frac{5}{2} nuT$ is needed because formulas valid for f in the electron frame are used. In a strictly one-dimensional (1-D) plasma one usually has $u = 0$ and then (5) determines q (and E_{nl}) in terms of the temperature gradient.

Here we use (4) to arrive at (5) with new expressions for the kernels in terms of one single integral,

$$J^*(\theta) = 8\pi^{1/2} \int_0^\infty ds\, s^{3/2} \exp\left(-s - \frac{\theta}{s^{3/2}} \right),$$

$I^* = 3J^* - 2\theta\, dJ^*/d\theta$, $\quad L^* = \frac{3}{4} I^* + \frac{1}{4} J^*$, and $K^* = 4L^* - 2\theta\, dL^*/d\theta$. All widths are around $\Delta\theta \simeq 10$; from $\xi_H/T'^2 \sim \Delta\theta$ we recover (1'). In the classical limit, $\xi_H \gg T'^2 \Delta\theta$, we only need the complete integrals of the kernels, which are equal for the new and old expressions, e.g., $\int_0^\infty d\theta (I^* - I) = 0$, so there is exact agreement with the result in Ref. 1, Spitzer's formula. In the opposite limit we just need the values at $\theta = 0$, slightly higher for the new kernels ($I^*/I = J^*/J \simeq 1.178, K^*/K = L^*/L \sim 1.123$ at $\theta = 0$), so we find a heat flux 12.3% above that in Ref. 1. The difference falls within the asymptotic accuracy of the model: the step from (3) to (4), consistent with the model itself, does not impair its accuracy.

Next we note the following fact: $\int_0^\infty K\, d\theta / \int_0^\infty L\, d\theta - \int_0^\infty I\, d\theta / \int_0^\infty J\, d\theta$ and $K(0)/L(0) - I(0)/J(0)$ have unity as a common value. The same applies, of course, to the new kernels. Consequently, the formula

$$q = \frac{-(\lambda_{ei}/\lambda_{ee})^{1/2}}{4\pi(3m_e)^{1/2}} \int dx'\, n' T'^{1/2} \frac{\partial T'}{\partial x'} L^*(\theta) \qquad (6)$$

exactly recovers results from Eq. (5) for *both* smooth (or classical) and steep limit profiles, and could be used for intermediate profiles, as a very convenient approximation.

For nonvanishing u we use the ion force on electrons, $R \equiv \int C_{ei} m_e v\mu\, d\bar{v}$ (a quantity of interest in itself), instead of the auxiliary field E_{nl}. The electron-momentum equation $R = neE + d(nT)/dx$, directly obtained from the kinetic equation, gives $eE_{nl} = R/n - \frac{7}{2} dT/dx$. We then make the change $I^*\, dT'/dx' + J^* eE'_{nl} \to (I^* - \frac{7}{2} J^*)(dT'/dx') + J^* R'/n'$, and similarly for K^*, L^* in (5). Though q and R are nonlocal they are linear in both u and dT/dx: we write $q = q_T + q_u$, $R = R_T + R_u$.[2] All results up to now correspond to q_T, with $u = 0$; in this case we obtain for R_T the classical value $-\frac{3}{2} n\, dT/dx$ in the limit $\xi_H \gg T'^2 \Delta\theta$, and $\frac{1}{2} n\, dT/dx$ in the opposite limit: note the change in sign. For $dT/dx = 0$ and the classical limit, one recovers Ohm's law and the known thermoelectric flux $q_u = \frac{3}{2} nuT$, meeting Onsager's principle, $q_u d \ln T/dx + R_T u = 0$; for $\xi_H \ll T'^2 \Delta\theta$, however, this principle would require

$$\frac{(\int dx'\, T'^{1/2} R'_u)}{\int dx'\, T'^{-1/2} R'_u} = \frac{2TJ^*(0)}{L^*(0)},$$

an equality that will not hold, in general.

Luciani and co-workers gave an early analysis of nonlocal heat flux for arbitrary Z_i. Physical arguments, and a fit to numerical simulations and the classical limit,[5,6] led them to an equation like (6) with a kernel

$$\bar{L}(\theta) \equiv 192\pi^{1/2} \frac{4}{31} \left(\frac{\ln \Lambda_{ei}}{6 \ln \Lambda_{ee}} \right)^{1/2} \times \exp\left[-\theta \frac{4}{31} \left(\frac{\ln \Lambda_{ei}}{6 \ln \Lambda_{ee}} \right)^{1/2} \right],$$

written here in present variables and for Z_i large; the ratio $\ln \Lambda_{ei}/\ln \Lambda_{ee}$ stems for their using a single Coulomb logarithm. They also gave an integrodifferential iterative procedure to determine f (its Legendre expansion broken at some order) and then q_T;[6] from the first iteration and setting $E = 0$, they derived (6) with a kernel, given graphically, quite close to $\bar{L}$. Referring to $E = 0$ as the isobaric case, they took $nT =$ const: note that this would imply $R_T = 0$. Bendib

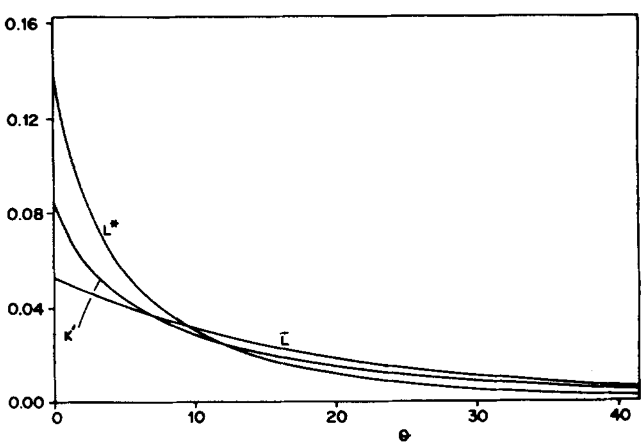

FIG. 1. Kernels for use in Eq. (6), L^*, $\bar{L}$ (Ref. 6), and K' (Ref. 8), vs $\theta \equiv T'^{-2} |\int_x^{x'} (6Z_i \ln \Lambda_{ee} \ln \Lambda_{ei})^{1/2} \pi e^4 n''\, dx''|$. For $\bar{L}$ we set $\ln \Lambda_{ee}/\ln \Lambda_{ei} = 1$.

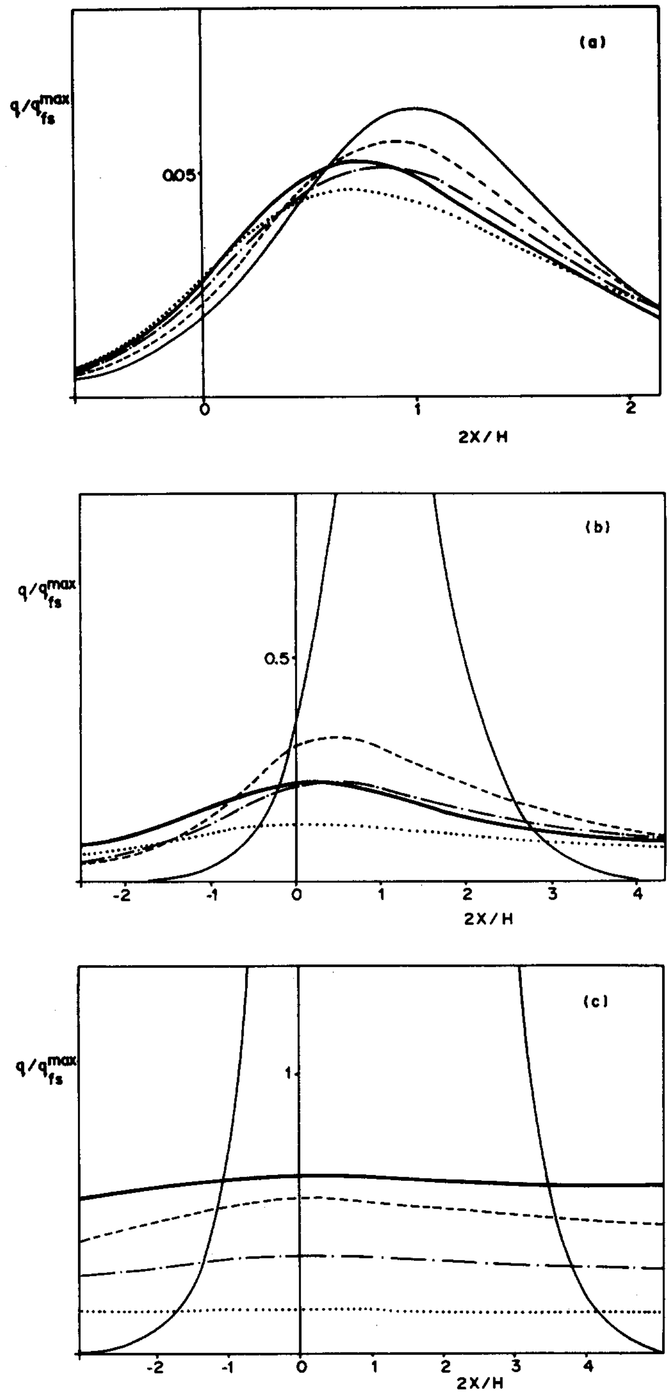

FIG. 2. Heat flux for fixed profiles $T_0/T = n/n_0 = 1 - \frac{1}{2}\tanh 2x/H$, $Z_i = 10$, and $Hn_0e^4\pi(6Z_i \ln \Lambda_{ee} \ln \Lambda_{ei})^{1/2}T_0^{-2}$, equal to (a) 400, (b) 20, and (c) 1. Flux normalized to a free-streaming value at the highest temperature, $2^{1/2}n_0T_0^{3/2}/m_e^{1/2}$; scheme (5), —; Spitzer, —; Eq. (6) with kernel L^*, ---; $\bar{L}$, ···; or K', -·-.

et al.[7] accounted for electric field effects in Luciani's model through the *ad hoc* change $\bar{L} \to \bar{L} \exp[(e\phi - e\phi')/T']$. For $e\phi$ they suggested using $e\,d\phi/dx = -T\,d\ln n/dx - \frac{5}{2}\,dT/dx$, an equation equivalent to always requiring $R_T = -\frac{3}{2}n\,dT/dx$.

Figure 1 compares $\bar{L}$ with L^*, setting $\ln \Lambda_{ei} = \ln \Lambda_{ee}$. Since $\bar{L}$ was fitted to recover the classical limit the area under the curve is the same, but differences are substantial. Also shown is $K' = (K^F - L^F I^F/J^F)^F$ with superscript F for Fourier transform, exact in the classical limit and also for small temperature variations, which Holstein and Decoster[8] considered as a kernel for (6) while examining nonlocal flux models.

We have numerically determined the heat flux for an infinite, static plasma with a given temperature profile, $T = T_0(1 - \frac{1}{2}\tanh 2x/H)^{-1}$, $nT = n_0T_0$, for which $\theta(x,x')$ is found explicitly. Figure 2 compares q_T as given by Spitzer's formula, the complete scheme (5), and Eq. (6), with L^*, $\bar{L}$, or K' for $Z_i = 10$ and values of $Hn_0e^4\pi(6Z_i \ln \Lambda_{ee} \ln \Lambda_{ei})^{1/2}T_0^{-2}$ = (a) 400, (b) 20, and (c) 1. All curves are in close agreement for case (a). For (b), lying at the heart of the regime of interest, all three kernels, L^*, $\bar{L}$, and K', especially the last one, give a reasonable approximation to scheme (5), well below Spitzer's result. For profile (c) only L^* remains valid. Summing up, if accuracy is preferred to convenience, the complete scheme must be used; otherwise one should use Eq. (6) with kernel L^*, which is simple and remains reasonably accurate throughout the range of validity of that scheme.

To conclude we note that Eq. (4) was derived by Luciani and co-workers[9] by a resummation procedure. They also gave evidence supporting the use of a two-term Legendre-polynomial expansion with $f_0 \neq f_M$, for arbitrary Z_i.[6] It would thus be possible to extend the formalism here continued from Ref. 1, which is strictly valid for Z_i large ($\lambda_{ee} \gg \lambda_{ei}$), to values $Z_i \sim 1$. One writes f for f_0 in C_{ee} as given at the beginning and adds to it a term C_{ei}/Z_i (already retained in Ref. 1).

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