Self-consistent, nonlocal electron heat flux at arbitrary ion charge number

Juan R. Sanmartín and J. Ramírez
E.T.S.I. Aeronáuticos, Universidad Politécnica, 28040 Madrid, Spain
R. Fernández-Feria
E.T.S.I. Industriales, Universidad de Sevilla, 41012 Sevilla, Spain
F. Minotti
Laboratorio de Fisica del Plasma, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina

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A single, nonlocal expression for the electron heat flux, which closely reproduces known results at high and low ion charge number Z, and "exact" results for the local limit at all Z, is derived by solving the kinetic equation in a narrow, tail-energy range. The solution involves asymptotic expansions of Bessel functions of large argument, and (Z-dependent) order above or below it, corresponding to the possible parabolic or hyperbolic character of the kinetic equation; velocity space diffusion in self-scattering is treated similarly to isotropic thermalization of tail energies in large Z analyses. The scale length H characterizing nonlocal effects varies with Z, suggesting an equal dependence of any *ad hoc* flux limiter. The model is valid for all H above the mean-free path for *thermal* electrons.

I. INTRODUCTION

The classical expression for the electron heat flux in a plasma defines the thermal conductivity through a local relation between flux and temperature gradient.[1–3] This local (Fourier) law was originally supposed to apply, as usual in kinetic theory, whenever the temperature scale length, $H \equiv |\nabla \ln T|^{-1}$ with T the electron temperature, was much larger than some thermal mean-free path for electron scattering λ_T. Mean-free paths in plasmas, however, are strongly energy dependent. As an easily verified consequence, electrons contributing most to the heat flux have energies well above thermal. This means in turn that the mean-free paths of interest are much longer than λ_T, thus explaining why classical flux calculations have been found to fail at H as large as $10^2\lambda_T$.

Albritton *et al.*[4] made good use of the energy dependence of mean-free paths in deriving a self-consistent formalism for the heat flux that has the nonlocal character exhibited by numerical calculations. In solving the Fokker–Planck equation they could let the flux-carrying (superthermal) electrons be non-Maxwellian, while main-body or thermal electrons were assumed Maxwellian. Further, by considering a large ion-charge number Z, they could make electron scattering by ions dominate self-collisions, and thus take those superthermal or tail electrons as nearly isotropic. Recently, Sanmartín *et al.*[5] have simplified the nonlocal results by noting that the flux-carrying electrons, having energies centered at $\epsilon_* \simeq 6.5T$, cover a *narrow* energy range, T. They also used a property of the kernels in the coupled integrals of the overall formalism to obtain a single formula for the heat flux, $\mathbf{q}_e$.

The first detailed model of nonlocal transport, by Luciani and co-workers,[6] considered Z arbitrary and used physical arguments in a fit to numerical simulations, later elaborated in a series of papers.[7,8] Prasad and Kershaw[9] and Epperlein and Short[10] showed that nonlocal models break down at very steep temperature profiles, when they may lead to heat flowing in the direction of the temperature gradient, $\mathbf{q}_e \cdot \nabla T > 0$. Note, however, that the regime $H \lesssim \lambda_T$ does not make sense in the formalism of Ref. 4, because then main body electrons would not be Maxwellian. Further, Sanmartín *et al.*[5] have showed that the formalism has a characteristic scale length $H \sim \lambda_T Z^{1/2}(\epsilon_*/T)^{3/2}$, failing ultimately at a much shorter H;[11] in practice, this reverts to the same failure condition, $H < \lambda_T$. Thus, a nonlocal model should *just* make collisional kinetic theory valid throughout the range $H > \lambda_T$, as originally expected from the local Fourier law. Recently, Ramírez and Sanmartín proved that including nonlocal transport in the self-similar expansion of a laser plasma does extend in a limited way the validity of a previous classical analysis.[12]

The extension of a self-consistent nonlocal formalism to values $Z \sim 1$ presents the main difficulty that self-collisions are then, in principle, as frequent as collisions with ions. There is broad evidence, however, supporting the idea that, even at low Z, scattering is the dominant process, the electron distribution function becoming isotropic faster than Maxwellian.[7,13] This fact allows us to use the same expansion scheme of the limit $Z \gg 1$,[4,5] but now retaining self-collisions at the lowest order, representing scattering. Minotti and Ferro-Fontan,[14] who considered self-scattering without diffusion in velocity space, found a hyperbolic equation for the anisotropic part of the distribution function, as opposed to the parabolic equation of the $Z \gg 1$ limit. They obtained heat flux results for low Z, which show good agreement with experimental data at nonlocal conditions and $Z = 1$. Their expressions, however, recover neither the nonlocal results at high Z nor the classical results of local conditions, at low Z. Murtaza *et al.* considered moderate Z, keeping terms of order $4/(Z+1)$, taken as a small parameter. They ignored diffusion in self-scattering at some point in their approximate scheme, and again failed to recover classical results at low Z.[15]

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In this work we use the narrow energy-range idea of Ref. 5 and asymptotic expansions of Bessel functions of large argument and *arbitrary* order, to develop a nonlocal formalism valid for all ion-charge numbers. Velocity diffusion in self-scattering is simplified following the procedure used in Ref. 4 for thermalization. We further follow Ref. 5 in reducing the overall formalism to a single formula for the heat flux. The next section gives the formulation of the problem. The electron distribution function and a nonlocal flux law, for arbitrary Z, are given in Secs. III and IV, respectively. Results are summarized in a final section.

II. MODEL FORMULATION

Let $f_e(\mathbf{v},\mathbf{r},t)$ be the electron distribution function and move to the electron frame by using $\mathbf{w}=\mathbf{v}-\mathbf{u}_e(\mathbf{r},t)$, where $\mathbf{u}_e$ is the mean directed velocity. Neglecting hydrodynamic effects in the kinetics on the basis of conditions $u_e \ll (T/m_e)^{1/2}$ and $\partial/\partial t \sim \mathbf{u}_e\cdot\nabla$, usual in quasineutral flows, the equation for $f_e(\mathbf{w},\mathbf{r})$ is[2]

$$\mathbf{w}\cdot\nabla f_e+\frac{e\,\nabla\phi}{m_e}\cdot\frac{\partial f_e}{\partial \mathbf{w}}=C_{ei}+C_{ee}, \qquad (1)$$

with $-\nabla\phi$ the electric field. We take collision terms C_{ei} *and* C_{ee} dominant (and f_e close to the local Maxwellian $f_{\rm M}$) at thermal energies, but not at the tail energies of interest, centered at some value $\epsilon_* \gg T$. Starting with $C_{es} (s=e,i)$ in a form equivalent to the Landau expression,[11]

$$C_{es}=\frac{2\pi e^2 e_s^2 \ln \Lambda_{es}}{m_e^2}\frac{\partial}{\partial \mathbf{w}}\cdot\left[\left(\frac{\partial f_e}{\partial \mathbf{w}}-\frac{m_e}{m_s}f_e\frac{\partial}{\partial \mathbf{w}}\right)\cdot\frac{\partial^2}{\partial \mathbf{w}\,\partial \mathbf{w}}\int|\mathbf{w}-\mathbf{w}'|f_s(\mathbf{w}')d\mathbf{w}'\right],$$

we neglect terms of order m_e/m_i, and first assume $\mathbf{u}\equiv\mathbf{u}_e-\mathbf{u}_i=0$ (no current), to obtain the usual (pure scattering) approximation for C_{ei},

$$C_{ei}=C'_{ei}(f_e)\equiv\frac{\partial}{\partial \mathbf{w}}\cdot\left(\frac{\partial f_e}{\partial \mathbf{w}}\cdot\frac{w^2\mathsf{I}-\mathbf{ww}}{2\tau_{ei}(w)}\right)\quad\left(\tau_{ei}\equiv\frac{m_e^2 w^3}{4\pi e^4 Zn \ln \Lambda_{ei}}\right),$$

vanishing for isotropic f_e ($\mathsf{I}\equiv$unit tensor). Rewriting C_{ee} (exactly) as

$$C_{ee}=\frac{w^3 \ln \Lambda_{ee}/\ln \Lambda_{ei}}{2\tau_{ei}Z}\left(\frac{\partial^2 f_e}{\partial \mathbf{w}\,\partial \mathbf{w}}:\frac{\partial^2}{\partial \mathbf{w}\,\partial \mathbf{w}}\int|\mathbf{w}-\mathbf{w}'|\times\frac{f_e(\mathbf{w}')}{n}d\mathbf{w}'+8\pi\frac{f_e^2}{n}\right), \qquad (2)$$

and noting that thermal electrons contribute most to the $\mathbf{w}'$ integral, we expand $|\mathbf{w}-\mathbf{w}'|$ in powers of the small ratio $w'/w\sim(T/\epsilon_*)^{1/2}$. With a two-term expansion,

$$\int|\mathbf{w}-\mathbf{w}'|\frac{f_{\rm M}(w')d\mathbf{w}'}{n}\simeq w\left(1+\frac{T}{m_e w^2}\right), \qquad (3)$$

and neglecting the last term in the bracket of (2), which is smaller than the first in the tail-to-thermal density ratio, we also get a linear approximation $C'_{ee}(f_e)$ for C_{ee}.

For $Z\gg 1$, C'_{ei}/C'_{ee} is large and f_e at energies around ϵ_* may be near-isotropic and yet not near-Maxwellian,

$$f_e=f_0(w)+\mathbf{w}\cdot\mathbf{g}(w), \qquad (4)$$

with $f_0\neq f_{\rm M}$ and $\mathbf{w}\cdot\mathbf{g}/f_0$ small. To dominant terms we set $f_e=f_0$ on the left of Eq. (1), ignore C'_{ee} on the right and obtain

$$\mathbf{w}\cdot\nabla f_0=C'_{ei}(\mathbf{w}\cdot\mathbf{g})=-\mathbf{w}\cdot\mathbf{g}/\tau_{ei}, \qquad (5)$$

having changed from variables $(w,\mathbf{r})$ to $(\epsilon\equiv\frac{1}{2}m_e w^2-e\phi,\mathbf{r})$. A velocity-angle average of (1) yields a second relation between f_0 and $\mathbf{g}=-\tau_{ei}\nabla f_0$,

$$\langle\mathbf{w}\cdot\nabla(\mathbf{w}\cdot\mathbf{g})\rangle=C'_{ee}(f_0)\equiv\frac{\ln \Lambda_{ee}}{\ln \Lambda_{ei}}\frac{m_e w^2}{\tau_{ei}Z}\frac{\partial}{\partial\epsilon}\left(f_0+T\frac{\partial f_0}{\partial\epsilon}\right). \qquad (6)$$

On the left we have dropped a term $2\tau_{ei}e\,\nabla\phi\cdot\nabla f_0/3m_e$, small by a factor $2e\phi/m_e w^2\sim T/\epsilon_*$ because the electron momentum equation,

$$0\simeq-\nabla(nT)+ne\,\nabla\phi+\mathbf{R}, \qquad (7)$$

with $\mathbf{R}$ the ion–electron friction, gives $e\phi\sim T$. We will write similarly $\epsilon+e\phi\simeq\epsilon$ for $\frac{1}{2}m_e w^2$ when appearing as a power, but not in a large exponent, as in the local Maxwellian, $f_{\rm M}=n(m_e/2\pi T)^{3/2}\exp[-(\epsilon+e\phi)/T]$.[16] Thus, for consistency, one should drop the term $T/m_e w^2$ in Eq. (3), i.e., the last term in (6), if f_0 followed a power law (small $T\partial \ln f_0/\partial\epsilon$), but should keep it if instead $f_0\simeq f_{\rm M}$ ($T\partial \ln f_0/\partial\epsilon\simeq-1$). At this point, one makes a crucial ansatz, $|T\partial \ln|f_0-f_{\rm M}|/\partial\epsilon|\ll 1$,[4] leading to

$$\nabla\cdot\mathbf{g}=\frac{3m_e}{\tau_{ei}Z_*}\frac{\partial(f_0-f_{\rm M})}{\partial\epsilon}\left(Z_*\equiv\frac{Z\ln \Lambda_{ei}}{\ln \Lambda_{ee}}\right). \qquad (6')$$

This amounts to using the classical limit, $f_0=f_{\rm M}$, in the $\partial^2 f_0/\partial\epsilon^2$ term of Eq. (6).

For $Z\sim 1$ there is no large parameter to allow writing Eq. (4). To the order considered, and for one-dimensional geometry, that approach, however, is equivalent to expanding f_e in Legendre polynomials P_n and neglecting $n\geqslant 2$ terms; there is evidence supporting this use of a two-term Legendre expansion, with $f_0\neq f_{\rm M}$, down to values $Z=O(1)$.[7,13] In a sense, this indicates that f_e becomes isotropic faster than Maxwellian, and allows writing $C'_{ei}+C'_{ee}$ (nonisotropic)$\gg C'_{ee}$ (isotropic). Thus, we keep (4) and (6′) and add to Eq. (5) the self-scattering term

$$C'_{ee}(\mathbf{w}\cdot\mathbf{g})=-\frac{\mathbf{w}}{\tau_{ei}Z_*}\cdot\left[\mathbf{g}\left(1-\frac{T}{m_e w^2}\right)-\frac{\partial}{\partial w}\left(w\mathbf{g}+T\frac{\partial}{\partial\epsilon}w\mathbf{g}\right)\right], \qquad (8)$$

where we may drop the term $T/m_e w^2$ against unity. Finally we now allow a current, $\mathbf{u}\neq 0$, by including in (5), the correction term

$$C_{ei}-C'_{ei}(\mathbf{w}\cdot\mathbf{g})\simeq\frac{\mathbf{w}\cdot\mathbf{u}}{\tau_{ei}w}\frac{\partial f_0}{\partial w}, \qquad (9)$$

where, from the ansatz preceding (6′), one uses $\partial f_{\rm M}/\partial w$ for $\partial f_0/\partial w$.

Take now gradients and current along the x axis and define

$$h \equiv w g_x,$$

Eqs. (6′) and (5), with the new terms (8) and (9), read as

$$w\tau_{ei} Z_* \frac{\partial h}{\partial x} = 6\epsilon \partial(f_0 - f_M)/\partial\epsilon, \tag{6''}$$

$$w\tau_{ei} \frac{\partial f_0}{\partial x} = \frac{2}{Z_*} \epsilon \frac{\partial}{\partial \epsilon}\left(h + T\frac{\partial h}{\partial \epsilon}\right) - \frac{Z_*+1}{Z_*} h + (2m_e\epsilon)^{1/2} u \frac{\partial f_M}{\partial \epsilon}. \tag{5'}$$

Once h (and f_0) are determined, one uses Eq. (7) and the equations

$$0 = \int f_e w_x \, d\mathbf{w} = \frac{8\pi}{3m_e^2} \int_0^\infty \epsilon h \, d\epsilon, \tag{10}$$

$$q_e = \int f_e \frac{1}{2} m_e w^2 w_x \, d\mathbf{w} = \frac{8\pi}{3m_e^2} \int_0^\infty \epsilon^2 h \, d\epsilon, \tag{11}$$

to obtain $\mathbf{R}$ and the heat flux $\mathbf{q}_e$ in terms of ∇T and $\mathbf{u}$. Equation (10) expresses the fact that the mean directed velocity of electrons vanishes in their own frame.[17] For consistency, we set the lower limit in the integrals (10) and (11) equal to zero instead of $-e\phi$.

The $\partial^2 h/\partial\epsilon^2$ term of Eq. (5′) was ignored in Ref. 14, where good agreement was found with experimental data at highly nonlocal conditions and $Z=1$. The low-Z results found there, however, disagree substantially with classical values in the collision-dominated limit; actually, it has been shown that to obtain transport coefficients close to classical, from an analysis of (5′) with $f_0 = f_M$, one must retain the $\partial^2 h/\partial\epsilon^2$ term.[11] Here, we do retain it, and lower the order of Eq. (5′), as in going from (6) to (6′), by writing $h + T\,\partial h/\partial\epsilon = h[1 + T\partial(\ln h)/\partial\epsilon]$ and using the classical value of the bracket. Note that this bracket, contrary to the expression $1 + T\,\partial(\ln f_0)/\partial\epsilon$ in Eq. (6), *does not vanish* in the collisional limit. It will be later shown that the effect of the term retained does decrease as one moves to nonlocal conditions.

Since a substantial departure of f_0 from a Maxwellian is being considered,

$$|f_0 - f_M| \sim f_M, \tag{12}$$

the ansatz leading from (6) to (6′) may be rewritten as

$$\left|\frac{\partial(f_0 - f_M)}{\partial\epsilon}\right| < \frac{f_M}{T} = \left|\frac{\partial f_M}{\partial \epsilon}\right|. \tag{13}$$

Conditions (12) and (13) can hold simultaneously only within a narrow energy range ($\sim T$), although this range may include the electrons carrying the heat flux. It will thus suffice to obtain $T\partial(\ln h)/\partial\epsilon$ (in the classical limit) at $\epsilon \simeq \epsilon_*$. To determine $\epsilon_*(Z)$ one maximizes $\epsilon^2 h(\epsilon)$ in the integral (11), and finds

$$\left.\frac{\partial(\ln h)}{\partial\epsilon}\right|_{\epsilon_*} = \frac{-2}{\epsilon_*}.$$

To get the classical form of h, we use Eq. (5′) with $u=0$ and $f_0 = f_M$. For $Z \to \infty$, one has

$$h(\text{classical}) \propto \epsilon^2(\tfrac{5}{2} + \beta_0 - \epsilon/T)\exp(-\epsilon/T),$$

where Eq. (7) and Braginskii's result, $\mathbf{R} = -\beta_0 n \nabla T$ ($\beta_0 = \frac{3}{2}$), are used, yielding the well-known value $\epsilon_* \simeq 6.56T$. For $Z=1$, Eq. (5′) at large ϵ/T gives[11]

$$h(\text{classical}) \propto \epsilon[4(1+\beta_0)(1+\epsilon/T) - 3\epsilon^2/T^2]\exp(-\epsilon/T),$$

with $\beta_0 \simeq 0.71$, yielding $\epsilon_* \simeq 5.93T$. To the accuracy considered in the analysis, and because self-scattering has vanishing effects at $Z \to \infty$, we may just set $\epsilon_*(Z) = 6T$. Thus, we approximate the first term on the right-hand side of (5′) in the form

$$\frac{2\epsilon}{Z_*} \frac{\partial(h + T\,\partial h/\partial\epsilon)}{\partial\epsilon} \simeq \frac{4\epsilon}{3Z_*} \frac{\partial h}{\partial \epsilon}. \tag{14}$$

III. THE ELECTRON DISTRIBUTION FUNCTION

Eliminating f_0 between Eqs. (6″) and (5′), where (14) is used, we obtain

$$\frac{\partial^2 h}{\partial \xi^2} + \frac{Z_*+1}{Z_*\epsilon} \frac{\partial}{\partial\epsilon}\left(\frac{h}{\epsilon^2}\right) - \frac{4}{3Z_*\epsilon} \frac{\partial}{\partial\epsilon}\left(\frac{1}{\epsilon}\frac{\partial h}{\partial\epsilon}\right) = S(\xi,\epsilon) \equiv \left(\frac{6}{Z_*}\right)^{1/2} \left[\frac{\partial}{\partial\xi} + \left(\frac{Z_* m_e u^2}{3\epsilon^3 T^2}\right)^{1/2}\right] \frac{f_M}{T\epsilon}, \tag{15}$$

where source terms of order T/ϵ have been neglected on the right-hand side, and

$$\frac{d\xi}{dx} \equiv \left(\frac{6}{Z_*}\right)^{1/2} \epsilon^2 / w\tau_{ei} = \pi e^4 (6Z \ln\Lambda_{ei} \ln\Lambda_{ee})^{1/2} n(x). \tag{16}$$

Equation (15) is hyperbolic,[14] except in the limit $Z_* \to \infty$, when it becomes parabolic.[4,5] To solve it introduce the Fourier transform

$$\tilde{h}(k,\epsilon) = (2\pi)^{-1/2} \int d\xi \, e^{ik\xi} h(\xi,\epsilon),$$

and define

$$F(k,y) \equiv \epsilon^{-2(1+\nu)} \tilde{h},$$

$$y \equiv (3Z_*/16)^{1/2} k\epsilon^2, \quad \nu \equiv (3Z_* - 5)/16.$$

The equation for F is then

$$\frac{d^2F}{dy^2} + \frac{1}{y}\frac{dF}{dy} + \left(1 - \frac{\nu^2}{y^2}\right) F = \frac{-\tilde{S}(k,\epsilon)}{k^2 \epsilon^{2+2\nu}}. \tag{17}$$

Since $\tilde{h} \propto y^{1+\nu} F$ must vanish as $\epsilon \propto y^{1/2} \to \infty$, the solution to (17) is

$$F(k,y) = -\int_y^\infty \frac{\pi y' \, dy'}{2k^2} \frac{\tilde{S}(k,\epsilon')}{\epsilon'^{2+2\nu}} \times [J_\nu(y) Y_\nu(y') - J_\nu(y') Y_\nu(y)],$$

with $y' \equiv (3Z_*/16)^{1/2} k\epsilon'^2$, and J_ν, Y_ν the Bessel functions of the first and second kind, and order ν. Then, the Fourier invert of $\tilde{h}$ is

$$h(\xi,\epsilon) = -\frac{3Z_*}{32} \int d\xi' \int dk\, e^{ik(\xi'-\xi)} \epsilon^{2(1+\nu)} \times \int_\epsilon^\infty \frac{\epsilon'\, d\epsilon'}{\epsilon'^{2\nu}} (J_\nu Y'_\nu - J'_\nu Y_\nu) S(\xi',\epsilon').$$

In the expression for $S(\xi',\epsilon')$ we write

$$f_M(\xi',\epsilon') = f_M(\xi',\epsilon) \exp\left[\frac{-\epsilon}{T'}\left(\frac{\epsilon'}{\epsilon} - 1\right)\right],$$

where $T' \equiv T(\xi')$. For a general profile $T(x)$, the superthermal condition $\epsilon \gg T$ leads to $\epsilon \gg T'$.[5] Then, only a narrow range of values, $\Delta\epsilon' \sim T'$, contributes to the ϵ' integral above; in particular, we might set $S(\xi',\epsilon') \simeq S(\xi',\epsilon)\exp[(\epsilon-\epsilon')/T']$.

With this energy range the characteristic "length" $\Delta\xi$ in Eq. (15) is

$$\Delta\xi \sim \epsilon_* T \quad (Z_* \sim 1); \tag{18a}$$

$$\sim \epsilon_*^{3/2} T^{1/2} \quad (Z_* \gg 1). \tag{18b}$$

We may now use $\Delta\xi$ to estimate the ratio between the second and first terms in the expression for $S(\xi,\epsilon)$ in Eq. (15). This ratio is order $(m_e u^2/3\epsilon_*)^{1/2}$ for $Z_* \sim 1$, and of order $(Z_* m_e u^2/3T)^{1/2}$ for large Z_*. Since we took u small against $(T/m_e)^{1/2}$, effects due to the current are small, except possibly for large Z_*, a limit for which they have been already discussed.[5] Hereafter we drop the u term in S and write

$$S(\xi,\epsilon) \simeq \left(\frac{6}{Z_*}\right)^{1/2} \left(\frac{\partial}{\partial\xi}\right) \frac{f_M}{T\epsilon}.$$

A further consequence of the fact that the energy range is narrow, is that the arguments of the Bessel functions are typically large. Using $k \sim 1/\Delta\xi$ and (18a) and (18b), we have

$$2y \sim \epsilon_*/T \quad (Z_* \sim 1),$$

$$\sim (Z_* \epsilon_*/T)^{1/2} \quad (Z_* \gg 1).$$

Assume for now that Z_* is also large. Then we may simplify the Bessel functions by using asymptotic expansions for large argument and order,[18]

$$J_\nu\left[\left(\frac{3Z_*}{16}\right)^{1/2} k\epsilon^2\right] Y_\nu\left[\left(\frac{3Z_*}{16}\right)^{1/2} k\epsilon'^2\right] - J_\nu\left[\left(\frac{3Z_*}{16}\right)^{1/2} k\epsilon'^2\right] Y_\nu\left[\left(\frac{3Z_*}{16}\right)^{1/2} k\epsilon^2\right]$$

$$\simeq 2\sin[p\nu(\zeta^2-1)^{1/2}]/\pi\nu(\zeta^2-1)^{1/2}, \quad \zeta > 1,$$

$$\simeq 2\sinh[p\nu(1-\zeta^2)^{1/2}]/\pi\nu(1-\zeta^2)^{1/2}, \quad \zeta < 1,$$

where

$$\zeta \equiv \left(\frac{3Z_*}{16}\right)^{1/2} \frac{k\epsilon^2}{|\nu|}, \quad p \equiv \left(\frac{\epsilon'}{\epsilon}\right)^2 - 1,$$

and the $\epsilon' \simeq \epsilon$ condition was used. The above sinusoidal and exponential behaviors correspond to the hyperbolic and parabolic character of Eq. (15) for $Z_* = 0(1)$ and $Z_* \gg 1$, respectively. Further, note that dominant terms in a Bessel function at large argument are the same for order large but less than the argument and for order about unity.[18] Thus, we may now relax the requirement $Z_* \gg 1$.

We then have

$$h = -\frac{3Z_*\epsilon^3}{32\pi} \int d\xi' \int dk\, e^{ik(\xi'-\xi)} \left(\frac{6}{Z_*}\right)^{1/2} \frac{\partial}{\partial\xi'} \times \frac{f_M(\xi',\epsilon)}{T'} \times I\left[\nu, \frac{\epsilon}{T'}, \left(\frac{3Z_*}{16}\right)^{1/2} \frac{k\epsilon^2}{|\nu|}\right],$$

$$I(\nu,s,\zeta) \equiv \int_0^\infty dp\, e^s \frac{\exp[-s(1+p)^{1/2}]}{\nu(1+p)^{\nu+1/4}} \times \begin{cases} \sin[p\nu(\zeta^2-1)^{1/2}]/(\zeta^2-1)^{1/2}, \\ \sinh[p\nu(1-\zeta^2)^{1/2}]/(1-\zeta^2)^{1/2}; \end{cases}$$

the expression above (below) applying for $\zeta > 1 (\zeta < 1)$. Since $s \equiv \epsilon/T'$ is large, we will take the limit $s \to \infty$ in evaluating the integral I. The exponent $\nu + \frac{1}{4}$ in its denominator resulted from writing $S(\xi',\epsilon') \simeq S(\xi',\epsilon^{1/2}\epsilon'^{1/2}) \times \exp[(\epsilon-\epsilon')/T']$, which is formally correct in that limit and improves the agreement with heat-flux values at classical conditions. We find

$$I = \left[\left(\frac{s}{2} + \frac{1}{4} + \nu\right)^2 + \nu^2\zeta^2 - \nu^2\right]^{-1}, \quad \zeta \gtrless 1.$$

Next, integrate by parts in ξ' and replace the derivative $\partial/\partial\xi'$ on $\exp[ik(\xi'-\xi)]$ by $-\partial/\partial\xi$. The k integral can then be carried out, to finally obtain

$$h = -\left(\frac{3}{2Z_*}\right)^{1/2} \int d\xi' \frac{T' f_M(\xi',\epsilon)}{\epsilon\sigma_z(\epsilon/T')} \frac{\partial}{\partial\xi} \exp\left[-\theta\sigma_z\left(\frac{\epsilon}{T'}\right)\right], \tag{19}$$

where

$$\theta = \frac{|\xi'-\xi|}{T'^2}, \quad \sigma_z(s) \equiv \left(\frac{4}{3Z_* s^2} + \frac{Z_*-1}{Z_* s^3}\right)^{1/2}. \tag{20}$$

Using now (19) in Eq. (5′), the isotropic part of the distribution function may be written as

$$f_0(\xi,\epsilon) \simeq \int d\xi'\, f_M(\xi',\epsilon) \frac{\sigma_z e^{-\sigma_z\theta}}{2T'^2}. \tag{21}$$

This is the same result of Ref. 5 if we write $\sigma_z = (T'/\epsilon)^{3/2}$, which is, in fact, the limit of σ_z as $Z_* \to \infty$. The characteristic range $\Delta\xi'$ in Eqs. (19) and (21), covering values ξ' below and above ξ, is $\Delta\xi' = 2T'^2/\sigma_z(T'/\epsilon_*)$ [compare with (18a) and (18b)]. Using Eq. (16), this gives the scale length $H \equiv \Delta\xi'\, dx'/d\xi'$ or

$$H = 2T^2/\pi e^4 n (6Z \ln\Lambda_{ei} \ln\Lambda_{ee})^{1/2} \sigma_z(T/\epsilon_*)$$

$$= (Z_* + \tfrac{11}{3})\lambda_T/(6Z_*)^{1/2} \sigma_z(T/\epsilon_*), \tag{22}$$

where we have introduced a representative total mean-free path for scattering of thermal electrons,

$$\lambda_T \equiv 2w\tau_{ei}(w)Z_*/(Z_*+\tfrac{11}{3}), \qquad \text{at} \quad \tfrac{1}{2}m_e w^2 = T.$$

Note that for H, as given by (22), Eq. (21) yields $f_0 \simeq f_M$ at thermal energies. Further, if H is much larger, (21) gives $f_0 \simeq f_M$ at energies $\epsilon \sim \epsilon_*$, to recover classical results for q_e. Finally, if f_M varies in a distance well below (22), changes in f_0 will lag behind, and condition (13) will not hold.[11]

The factor $(Z_*+\frac{11}{3})/Z_*$ in λ_T, representing electron–electron effects in scattering, is obtained by carrying the approximation (14) to its end, so that, in the local limit and for $u=0$, Eq. (5′) reads as $w\tau_{ei}\partial f_M/\partial x = -(Z_*+\frac{11}{3})h/Z_*$. This provides a rough way to correct for finite-Z effects, so that one might write $q_e(Z)/q_e(Z \gg 1)|_{\text{clas}} \simeq Z_*/(Z_*+\frac{11}{3})$. The factor $Z_*/(Z_*+\frac{11}{3})$ compares well with the factor $(Z_*+0.24)/(Z_*+4.2)$ obtained from an *ad hoc* fit to classical results,[1-3] and often used to correct the pure ion-scattering (high Z) heat flux.[10]

IV. THE HEAT-FLUX FORMULA

Use of (19) in Eqs. (10) and (11) determines q_e and R in terms of the temperature gradient. Equation (7) allows us to eliminate the electric potential, which appears in the local Maxwellian, $f_M \propto \exp[-(\epsilon+e\phi)/T]$.[5] Here, however, we will follow the formalism of Ref. 4 for the large Z_* case, and ignore (7); Eqs. (10) and (11) are then taken to give q_e and the electric potential or, for convenience, a "nonlocal" potential defined by

$$\frac{e\,\partial\phi_{\text{nl}}}{\partial x} \equiv \frac{e\,\partial\phi}{\partial x} - \frac{T\partial \ln n}{\partial x} + \frac{5}{2}\frac{\partial T}{\partial x}.$$

Since the expression for f_0 has a form similar to that in Ref. 5, we may directly write results similar to those of that paper.

The final expressions are thus

$$\{0,q_e\} = \int \frac{dx'\, n'\{1,T'\}}{4\pi(3m_e Z_* T')^{1/2}} \times \left(\{I_z^*,K_z^*\}\frac{\partial T'}{\partial x'} - \{J_z^*,L_z^*\}\frac{e\partial\phi'_{\text{nl}}}{\partial x'} \right). \tag{23}$$

The kernels I_z^*, J_z^*, K_z^*, and L_z^* are given as functions of both $\theta \equiv |\xi' - \xi|/T_e'^2$ and Z_* in terms of two simple integrals:

$$\{J_z^*(\theta),L_z^*(\theta)\} = 8\pi^{1/2}\int_0^\infty ds \exp[-s-\theta\sigma_z(s)]\frac{\{1,s\}}{\sigma_z(s)},$$

$$I_z^* = 3J_z^* - \frac{2\theta\, dJ_z^*}{d\theta}, \tag{24}$$

$$K_z^* = 4L_z^* - \frac{2\theta dL_z^*}{d\theta}.$$

Since the expression for $\sigma_z(s)$ in (20) goes into $s^{-3/2}$ as $Z_* \to \infty$, we exactly recover the results of Ref. 5 for that limit.

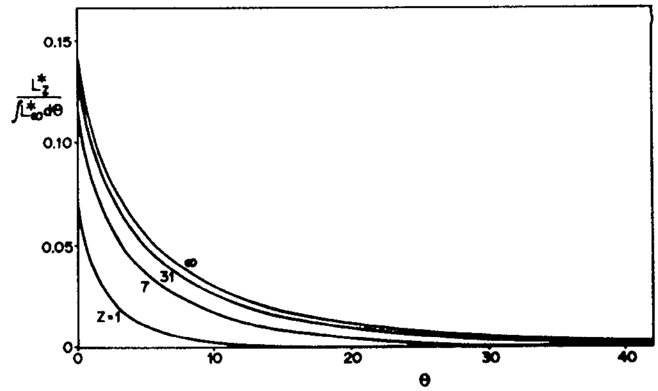

FIG. 1. Kernel $L_z^*(\theta)$ in the present heat-flux model, Eq. (25), normalized with $\int L_\infty^*(\theta)\, d\theta$, for several values of ion charge number Z.

In the classical case (smooth gradients) only the complete integrals of the kernels, e.g., $\int_0^\infty I_z^*(\theta)d\theta$, enter the results. In the opposite limit (steep gradients) only the values of the kernels at $\theta=0$ count. Further, the following relations:

$$\frac{\int_0^\infty K_z^*(\theta)d\theta}{\int_0^\infty L_z^*(\theta)d\theta} - \frac{\int_0^\infty I_z^*(\theta)d\theta}{\int_0^\infty J_z^*(\theta)d\theta} = \frac{K_z^*(0)}{L_z^*(0)} - \frac{I_z^*(0)}{J_z^*(0)} = 1,$$

the limit case of which for $Z_* \to \infty$ was given elsewhere,[5] can be easily shown to hold. One may then verify that a *single* expression,

$$q_e = -\int \frac{dx'\, n' T_e'^{1/2}}{4\pi(3m_e Z_*)^{1/2}}\frac{dT_e'}{dx'}L_z^*(\theta), \tag{25}$$

where L_z^* is given in (24),

$$L_z^* \equiv 8\pi^{1/2}\int_0^\infty s\, ds \exp\frac{[-s-\theta\sigma_z(s)]}{\sigma_z(s)},$$

and Z_*, σ_z, and θ (and ξ), are given in Eqs. (6′), (16), and (20), recovers exactly results from the coupled equations (23) for both smooth and steep temperature profiles, and can be a convenient approximation for intermediate profiles, as discussed in Ref. 5 for large Z_*. A plot of $L_z^*(\theta)$, normalized with $\int_0^\infty L_\infty^*(\theta)d\theta$, is given in Fig. 1 for some values of Z_*.

For smooth enough gradients, Eq. (25), or Eqs. (23), yield

$$q_e = -\frac{3\gamma_0(Z)T^{5/2}\,\partial T/\partial x}{4(2\pi)^{1/2}m_e^{1/2}e^4 Z \ln\Lambda_{ei}}, \tag{26}$$

with γ_0 given by

$$\gamma_0(Z) \equiv \int_0^\infty 2L_z^*(\theta)\frac{d\theta}{9\pi^{3/2}}. \tag{27}$$

Equation (26) is the usual expression for the classical flux. Numerical results on γ_0 by Braginskii[2] were extended, and confirmed within about 1%, by Epperlein and Haines;[3] earlier, Spitzer and Härm[1] had numerically given

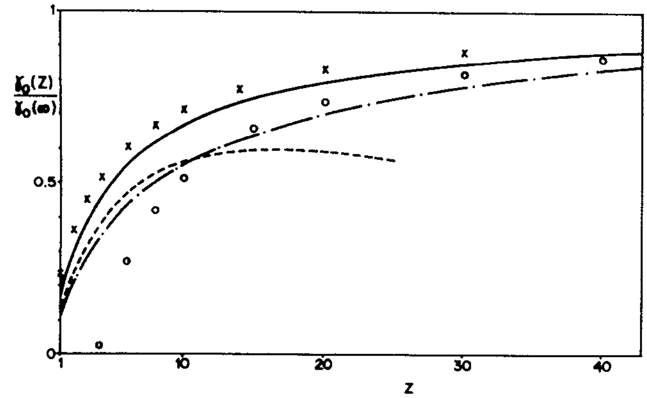

FIG. 2. Coefficient $\gamma_0(Z)$, normalized with its value for $Z\rightarrow\infty$, in the classical flux law, Eq. (26): Exact results (Refs. 1–3), ×; present model, Eq. (27′), ——. Also shown are results from a crude variant of the model (see the end of Sec. IV), –·–, and from Refs. 14, – – –, and 15, ○.

$3\pi\gamma_0/320\equiv\delta_T(Z)\epsilon(Z)$, in terms of two other coefficients, δ_T and ϵ. Figure 2 compares our analytical formula (27), i.e.,

$$\gamma_0=\frac{16}{9\pi}\int_0^\infty \frac{se^{-s}\,ds}{\sigma_z^2(s)}=\frac{4Z_*}{3\pi}\int_0^\infty \frac{s^4e^{-s}\,ds}{s+3(Z_*-1)/4}, \tag{27′}$$

with the "exact" values.[1–3] For the comparison we set $\ln\Lambda_{ee}=\ln\Lambda_{ei}$, or $Z_*=Z$. The agreement is quite reasonable.

Also shown in Fig. 2 are results from a simpler variant of our model in which, instead of making use of (14), one just drops the $\partial^2h/\partial\epsilon^2$-term in Eq. (5′). This amounts to using a modified function $\sigma_z^2\equiv(2/Z_*s^2)+(Z_*-1)/Z_*s^3$. The variant model follows correctly the Z dependence of γ_0 but gives values too small, particularly at low Z, where it agrees with the results from Ref. 14, also shown in the figure together with the results from Ref. 15.[19] At extreme nonlocal conditions, we have $q_e\propto L_z^*(0)\propto\int_0^\infty s\times\exp(-s)ds/\sigma_z(s)$, and the variant produces weaker changes.

V. SUMMARY AND CONCLUSIONS

We have derived from the kinetic equation a nonlocal formalism for the heat flux valid for all Z. We recover results from Refs. 4 and 5 at large Z, and results similar to those in Ref. 14 at low Z. In the classical limit we obtain a thermal conductivity that follows closely the entire Z dependence known from "exact" calculations. We have further shown that, as in the large Z case, a single integral expression for q_e, Eq. (25), reproduces exactly results from the coupled equations of the entire formalism, for both smooth and steep profiles, and would be a very convenient approximation for intermediate profiles.

The kinetic equation could be solved because the energies of interest cover a narrow range, $\Delta\epsilon\sim T$, centered at a value $\epsilon_*\gg T$. The solution then involves asymptotic expansions of Bessel functions of large argument and Z-dependent order above or below the argument. The resulting exponential or sinusoidal behavior corresponds to the parabolic or hyperbolic character of the kinetic equation. To simplify the diffusion in velocity space, in the anisotropic self-collision term, we introduced the classical limit behavior, in a spirit similar to that taken in Ref. 4 to deal with thermalization. The model is accurate to order T/ϵ_*, or about 16%, similar to the accuracy of Coulomb logarithms.

We have shown that nonlocal transport is characterized by a scale length H given by Eq. (22). Note that the characteristic ratio H/λ_T is large and increases with Z. The breakdown of local transport at $H/\lambda_T\gg1$, instead of $H/\lambda_T\sim1$ as originally expected, explains why the limiter f in the flux bound $q_e<fnT^{3/2}/m_e^{1/2}$ of standard numerical simulations has been found to be small. If H/λ_T is well below the value (22), or $H\sim\lambda_T$ (when the distribution function ceases to be near-Maxwellian at thermal energies), nonlinear transport breaks down. In practice, one can write its range of validity as $1<H/\lambda<\infty$. For H below λ_T, one might resort back to the use of a limiter f with a Z dependence, $f\propto Z_*^{1/2}\sigma_Z(T/\epsilon_*)/(Z_*+\frac{11}{3})$, where $\epsilon_*\simeq6T$.

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[17] If there is current ($\mathbf{u}\neq 0$) but no self-scattering ($Z\rightarrow\infty$), one may shift to the ion frame, $\mathbf{w}^*\equiv\mathbf{v}-\mathbf{u}_i(\mathbf{r},t)$, where $C_{ei}\simeq C'_{ei}$ still holds, and use Eqs. (5) and (6′) with $\mathbf{w}\rightarrow\mathbf{w}^*$. At the end, one uses $n\mathbf{u}=\int\mathbf{w}^*f_e(\mathbf{w}^*)d\mathbf{w}^*$, $\mathbf{q}_e=\int\frac{1}{2}m_e w^{*2}\mathbf{w}^*f_e(\mathbf{w}^*)d\mathbf{w}^*-\frac{5}{2}n\mathbf{u}T_e$.

[18] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, DC, 1972), Sec. 9.3.

[19] Classical results from Ref. 15 can be written as $\gamma_0(Z)=Z\gamma_0(\infty)(3Z^2-16Z+17)/(Z+1)(3Z^2-2Z-5)$.