Interspecies transfer of momentum and energy in disparate-mass gas mixtures

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(Received 29 January 1986; accepted 18 September 1986)

The collision integrals describing the rate of exchange of momentum and tensorial energy between the components in a binary mixture of neutral gases with very different atomic masses are determined for arbitrary values of their two temperatures and velocities, for realistic intermolecular potentials, and allowing for large departures of the heavy gas from equilibrium conditions. In the range of interest where the system is perturbed within times of the order of the slow relaxation time characterizing the transfer of energy between unlike molecules, the light gas distribution function is Maxwellian to lowest order, with corrections given asymptotically in powers of the small parameter \( m/m_p \) formed with the ratio of the species molecular masses. Also, provided that the ratio \( T_p/T \) between the temperatures of the two gases remains much smaller than \( m_p/m \), the desired collision integrals may be evaluated asymptotically in powers of \( m/m_p \) in all generality. The computation is carried out in detail for the case when the interaction between atoms is described by a Lennard–Jones potential. A combination of numerical computations with optimal matching of analytical expressions valid for large and small slip velocities leads to a set of compact formulas which hold for the limits of high and low temperatures and to a general approximate expression for all temperatures.

I. INTRODUCTION

The relative inefficiency of the interchange of kinetic energy in collisions between gaseous molecules of very different masses makes it often necessary to describe their dynamical behavior in terms of two-fluid theories rather than by means of the ordinary equations of hydrodynamics. One important feature of two-fluid theories is the existence of a transfer of momentum and energy between the two species, so that these moments are not conserved for each gas, but only for the mixture as a whole. The form of the coupling terms measuring the rates of momentum and energy transfer has been discussed in the literature in a diversity of works concerned about such topics as free jet expansions,\(^1\) aero-
dynamic separation of species,\(^3\) reactive crossed molecular beam experiments,\(^4\) etc. A general treatment for the case when the two gases are under near-equilibrium conditions is given by Burgers.\(^5\)

The set of two-fluid governing laws may be obtained straightforwardly by taking moments in the Boltzmann equations describing the evolution of the velocity distribution functions \( f \) and \( f_p \) for the light and heavy gas, respectively. The following equations are obtained for the heavy species:

\[
\begin{align*}
\partial_t \rho_p + \nabla \cdot (\rho_p \mathbf{u}_p) &= 0, \\
\partial_t (\rho_p \mathbf{u}_p) + \nabla \cdot (\mathbf{P}_p + \rho_p \mathbf{u}_p \mathbf{u}_p) &= -\rho_p \mathbf{b}, \\
\partial_t \mathbf{P}_p + \nabla \cdot (2 \mathbf{Q}_p + \rho_p \mathbf{u}_p \mathbf{u}_p) + (\mathbf{P}_p \cdot \nabla) \mathbf{u}_p + \left[ (\mathbf{P}_p \cdot \nabla) \mathbf{u}_p \right]^T &= -\rho_p (\mathbf{E} + \mathbf{E}_{\text{self}}),
\end{align*}
\]

where the superscript \( T \) denotes the transposed tensor \( (A^T)^{ij} = A_{ji} \) and the hydrodynamic magnitudes, density, mean velocity and pressure tensor \( \rho_p, \mathbf{u}_p, \mathbf{P}_p \) are defined based on the mean speed of each individual species rather than the mean mixture velocity:

\[
\begin{align*}
\rho_p &= \int m_p f_p \, d^3 \mathbf{u}_p, \\
\rho_p \mathbf{u}_p &= \int m_p \mathbf{u}_p f_p \, d^3 \mathbf{u}_p, \\
\mathbf{P}_p &= \int m_p (\mathbf{u}_p - \mathbf{U}_p) (\mathbf{u}_p - \mathbf{U}_p) f_p \, d^3 \mathbf{u}_p, \\
Q_p &= \frac{1}{2} \int m_p (\mathbf{u}_p - \mathbf{U}_p) (\mathbf{u}_p - \mathbf{U}_p) (\mathbf{u}_p - \mathbf{U}_p) f_p \, d^3 \mathbf{u}_p.
\end{align*}
\]

The same definitions, without subscript, apply to the light gas. Following Burgers,\(^5\) the coupling terms \( \mathbf{b} \) and \( \mathbf{E} \) can be written as

\[
\begin{align*}
\rho_p \mathbf{b} &= \int m_p f_p (\mathbf{u}_p) d^3 \mathbf{u}_p \mathbf{B}, \\
-\rho_p \mathbf{E} &= \int m_p f_p (\mathbf{u}_p) d^3 \mathbf{u}_p (1 - (\mathbf{u}_p - \mathbf{U}_p) \mathbf{B} - (\mathbf{u}_p - \mathbf{U})),
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{B}(\mathbf{u}_p) &= M \int d^3 g g \sigma_1 f(\mathbf{u}_p - \mathbf{g}), \\
\Pi(\mathbf{u}_p) &= M^2 \int d^3 g \left( \frac{1}{2} (g^T - 3gg) \sigma_2 + 2gQ_1 \right) f(\mathbf{u}_p - \mathbf{g}),
\end{align*}
\]

\( \sigma_1 \) and \( \sigma_2 \) are the usual collision cross sections given in terms of the differential scattering cross section \( \sigma(\partial g) \) for the heavy–light collisions,

\[
Q_1 = 2\pi \int \sigma(\partial g) (1 - \cos \theta) \sin \theta \, d\theta \, d\varphi,
\]

\( \mathbf{g} \) is the relative molecular velocity \( \mathbf{u}_p - \mathbf{u} \), and \( M \) is the small
parameter formed with the masses \( m \) and \( m_p \) of the light and heavy gas,
\[
M = m/(m + m_p).
\]

Notice that we have written Eq. (3) for all the components of the stress tensor of the heavy gas. Contracting it with the unit tensor would yield the conservation equation for the thermal energy \( e \), free from the term \( E_{\text{self}} \) measuring the rate of change of \( P_p \) caused by heavy-heavy collisions. Only the trace of \( E \) would be of interest in standard two-fluid theories. However, here we will evaluate the full tensor \( E \) because of its importance in the case where the heavy gas is dilute and departs from equilibrium.

As can be seen from Eqs. (8)–(11), the evaluation of \( b \) and \( \Pi \) requires the specification of both velocity distribution functions, as well as the interaction potential between heavy and light molecules, needed to determine the differential scattering cross section.

Different choices for the interaction potential have been used in the literature for neutral gases. Inverse fourth order attractive potentials (i.e., Maxwellian molecules) have been used in a number of works, because the resulting \( Q \) vary as \( g^{-1} \) making expressions (8) and (9) independent of the velocity distribution functions. This assumption, however, leads to an unrealistic description of the transfer terms. A long range repulsive interaction decreasing as the inverse sixth power of the distance has been used in an attempt to give a simpler description of the behavior at low temperatures, dominated by the long range attractive part of the interaction. A full Lennard–Jones potential has been used by Patch among others.

When both species are close to equilibrium, the velocity distribution functions can be described by means of the Maxwellian expression to lowest order, or by the Chapman–Enskog solution to the Boltzmann equation to include first-order corrections in the gradients of the hydrodynamical quantities. Some of these corrections are discussed by Burgers, using Grad’s 13 moment method. A common procedure to describe nonequilibrium effects is to use an “ellipsoidal Maxwellian” function, which allows for different parallel and perpendicular temperatures. This function is suggested by experimental measurements of the parallel and perpendicular components of molecular speeds of pure gases or mixtures expanding into a vacuum. Since there is no theoretical foundation for the ellipsoidal function (though it may be produced for the heavy gas on the basis of the present work), the range of validity cannot be assessed. Other approximate fitting expressions for the velocity distribution, such as the drifting Maxwellian or the Hermite polynomial expansion model, are discussed by Patch.

Even when the distribution functions \( f \) and \( f_p \) are assigned, some other hypotheses can help to simplify the task of performing the coupling integrals. The one adopted most widely consists in linearizing, assuming small mean velocity or temperature differences between the two species; most of the published results are subject to this restriction. In free jet expansions, when ellipsoidal Maxwellian distribution functions are used, the large parallel to transversal temperature ratio can provide further simplifications. Schwartz and Andres relaxed the small slip velocity assumption obtaining an interpolation formula for the momentum coupling term for the case of an inverse sixth attractive potential [see discussion following Eq. (56)]. Burgers reports the momentum and energy (scalar) collision integrals for unrestricted velocity differences and unrestricted temperatures in the case of hard elastic spheres and Maxwellian distribution functions.

The present work is motivated by the conditions of our own experimental research and those of many situations of potential industrial interest involving disparate-mass gas mixtures. Typically the light gas is close to equilibrium, while the heavy gas is characterized by a narrow but far from Maxwellian distribution function. Furthermore, the slip velocity is not necessarily small, and the temperatures are such that both the repulsive and the attractive parts of the interaction potential may be important. We shall therefore use the Lennard–Jones interaction potential and will abstain from linearizing in the velocity slip. The standard near-equilibrium road will be used for the assignment of the distribution function \( f \) only, while \( f_p \) will be described following Bernstein and Fernandez de la Mora in taking advantage of the large disparity in the masses \( m \) and \( m_p \) between the two species to develop a systematic perturbation theory in powers of the small parameter \( m/m_p \). Our final expression for the integrals (8) and (9) will be valid to lowest order in \( m/m_p \), although the procedure to incorporate higher-order corrections is sketched. Our results will be limited by the condition that the molar fraction of the heavy gas does not exceed order unity \( (n_p \ll n) \); but such a constraint is satisfied in most situations in which disparate-mass mixtures are of practical relevance.

II. EXPRESSIONS FOR THE INTERSPECIES MOMENTUM AND ENERGY TRANSFER

One of the objectives of this work is to give easily used closed form expressions for the interspecies momentum and energy transfer terms shown in Eqs. (2) and (3). The expressions given in Eqs. (8)–(11) are completely general. Following Bernstein, the large mass ratio can now be exploited by noticing that, in the particular case where \( m \ll m_p \), the characteristic range of values of the heavy gas thermal speed \( u_p \equiv u_p - U_p \) is smaller than that of the light gas \( u \equiv u - U \) by a factor \( (T_p m/T_p m_p)^{1/2} \), which is much smaller than unity provided that the heavy species temperature does not become much larger than \( T \). As a result, the integration of Eqs. (8) and (9) over the \( u_p \) variable is straightforward: \( f_p \) is so narrowly centered at \( u_p = U_p \) (nearly a Delta function) that \( B \) and \( \Pi \) can only contribute to the integrals (8) and (9) in a small neighborhood of \( u_p = U_p \). We may thus approximate these functions within the integrand by their Taylor expansion about \( u_p = 0 \) (see Refs. 9 and 10):

\[
B(u_p) = (1 + u_p \cdot \nabla u_p) + \frac{1}{2} (u_p \cdot \nabla u_p) \cdot \nabla u_p + \ldots \cdot B(U_p),
\]

and analogously for \( \Pi(u_p) \). The integration over \( u_p \) can now be easily performed:

\[
\rho_p b = (\rho_p + \frac{1}{2} P_p \cdot \nabla u_p \cdot \nabla u_p + \cdots) B, \tag{14a}
\]

\[
-\rho_p E = (\rho_p + \frac{1}{2} P_p \cdot \nabla u_p \cdot \nabla u_p + \cdots) \Pi - (P_p \cdot \nabla \cdot B)^+, \tag{14b}
\]
where for a general square mixture the notation \( A^+ \equiv A + A^T \) is followed, and the right-hand side terms are evaluated at \( u_p = U_p \). As we see, \( f_p \) enters into Eqs. (14) only through its moments \( \rho_p, U_p, P_p, \) \( \cdots \). To first order in \( u_p^2 \), Eq. (13) is equivalent to the assumption that the heavy gas distribution function is so narrow that it can be considered a delta function centered in \( U_p \). In this approximation, \( u_p \) is simply substituted by \( U_p \) in Eqs. (10) and (11). The first correction, proportional to \( P_{p} \), takes into account the finite width of \( f_p \) and is proportional to the curvature of the \( B \) and \( \Pi \) functions at \( u_p = U_p \).

A specification of \( f \) is now necessary to proceed. Remarkably enough, within the range of parameters of interest, it is possible to determine \( f \) by a rigorous expansion in powers of the small quantity \( m/m_p \) (see Ref. 10).

\[
f = f_0\left[1 + O(m/m_p)\right],
\]

where \( f_0 \) is the Maxwellian distribution with number density \( n = \rho/m \) of mean velocity \( U \) and temperature \( T \). Equation (15) results from the fact that the departures of \( f \) from Maxwellian are of the order of the Knudsen number \( Kn \), defined as the ratio between \( \tau_1 \) (self-collision relaxation time given by the ratio \( \mu/p \) between its viscosity component and partial pressure) and \( t_f \) (fluid dynamic deceleration time \( \sim |U|/|D U/|DT| \)). Furthermore, if \( \tau \) is the slow relaxation time characteristic of the transfer of energy between two gases, by purely mechanical considerations one has that

\[
\tau/\tau_1 \sim m_p/m,
\]

in the range where \( n_p/n \) is up to order 1. In other words, the relative heavy molecule change of momentum or energy per collision with a light molecule is only a small fraction \( m/m_p \) of the relative change in a light gas self-collision.

The point of interest here is that the peculiar relaxation phenomena that make disparate-mass mixtures so interesting do set in when the system is perturbed within a characteristic macroscopic time \( t_f \) of the order of the slow relaxation time \( \tau \). Defining therefore the interspecies Knudsen number

\[
S = \tau/t_f,
\]

which is analogous to the Stokes number arising in aerosol dynamics, it results that \( S \) will take values of order 1 in the problems that occupy our attention. This same condition,

\[
S \sim 1,
\]

is what makes two-fluid theories really necessary, since ordinary hydrodynamics would be perfectly adequate to describe the region where \( S \ll 1 \). Combining now Eqs. (17) and (18) we conclude that

\[
Kn \sim mS/m_p \sim 1.
\]

Equation (16) is thus confirmed and the light gas distribution function can be considered Maxwellian with errors of the order \( m/m_p \):

\[
f(u) = n(m/2\pi kT)^{3/2} \exp\left(-|u - U|/2kT\right) \times \left[1 + O(m/m_p)\right].
\]

Higher-order corrections to Eq. (20) can be carried out systematically in the whole range of concentrations, and will be considered elsewhere.

Using the Maxwellian approximation (20), the differentiation with respect to \( U_p \) in (13) becomes equivalent to a differentiation with respect to \( \xi = U_p - U \). At the same time we notice that \( B, \Pi_1, \) and \( \Pi_2 \) (where \( \Pi_1 \) and \( \Pi_2 \) are the contributions of \( Q_1 \) and \( Q_2 \) to \( \Pi \)) are not independent:

\[
\Pi_1 = 2M\left[(kT/m)\nabla_\xi B + \xi B\right],
\]

although Eq. (21) is no longer true when higher-order corrections are incorporated in Eq. (20) for \( f \). The ensuing expressions are therefore only valid for the first approximation which takes \( f \) to be Maxwellian.

In order to determine \( B, \Pi_1, \) and \( \Pi_2 \), the interaction potential between light and heavy molecules has to be specified. Our choice is the Lennard–Jones potential

\[
\varphi(r) = 4\epsilon\left[(\sigma/r)^{12} - (\sigma/r)^6\right],
\]

where \( \epsilon \) and \( \sigma \) are the depth and radial location of the potential well. These two parameters are used to define the following dimensionless variables:

\[
T^* = kT/m\epsilon,
\]

\[
T_p^* = kT_p/\epsilon,
\]

\[
v = \xi/(2kT/m)^{1/2},
\]

\[
x = g/(2kT/m)^{1/2},
\]

where \( M_p = m_p/m_0, m_0 = m + m_p, \) and \( T_p \) is the tensor of temperatures \( T_p \equiv m_0P_p/kP_p ). Notice a slight variation in the normalization of \( T^* \) with respect to the expression \( T^* = kT/\epsilon \) given by Hirschfelder et al. \(^{11}\) The difference results from the product of two Maxwellian functions (one for each component) present in their treatment, whereas only one Maxwellian function is involved here because \( f_p \) is introduced by perturbation around a Delta rather than a Gaussian function. If terms of order \( m/m_p \) are neglected, the difference between the two normalizations disappears, but definition (23) must be used in a higher order theory. Before substituting the normalized variables, the integration in Eqs. (10) and (11) can be carried out over all the \( g \) directions by choosing the polar axis in the direction of \( \xi \), that is, \( e_3 = \xi/\xi \).

Then

\[
d^2g = g^2 dg d\Omega_\xi,
\]

where \( d\Omega_\xi \) is the solid angle differential. Integrating by parts over \( d\Omega_\xi \) leads to the following normalized expressions which depend only on three scalar functions:

\[
\Pi^* = \left[I_B(e_1e_1 + e_3e_3) - I_B e_3e_3\right] T^{*3/2},
\]

\[
\Pi^* = (1 - 3e_3e_3)I_{H1} T^{*3/2}/6,
\]

\[
B^* = T^*I_B e_3,
\]

where \( \Pi^* = \Pi_1/K_{H1} \) and \( B^* = B/K_B \), with

\[
K_B = 8\pi^{1/2}\sigma^2/m_p, \quad K_{H1} = 2(2emmax^{-1/2})^{1/2}K_B,
\]

and the following integrals have been introduced:

\[
I_B = \int_0^\infty x^4e^{-x^2}Q^*(xT^{*1/2})\left(\frac{\cos\alpha}{\alpha} - \frac{\sin\alpha}{\alpha^2}\right)dx,
\]

\[
I_{H1} = \int_0^\infty x^3e^{-x^2}Q^*(xT^{*1/2})\left(\frac{\sin\alpha}{\alpha} - \frac{3\cos\alpha}{\alpha^2} + \frac{3\sin\alpha}{\alpha^3}\right)dx.
\]
\[ I_{n} = \frac{1}{2} \frac{\partial I_{n}}{\partial v} + v I_{n}, \]  
and \( \alpha = 2u. \) Expression (33) giving \( I_{n} \) as a function of \( I_{n} \) is only convenient when using the analytic approximate limits of the \( I \) integrals determined below. When a numerical approach is adopted, it is better to use the full expression of \( I_{n} \), as an integral over \( x \) to avoid numerical differentiation.

Substituting these results into Eqs. (14), (15) and denoting \( b^{*} = b/K_{n} \) and \( E^{*} = E/K_{n} \), we obtain

\[ b^{*} = (1 + \frac{1}{2} M T^{* -1} T_{p} \nabla, \nabla, \nabla, \cdots) B^{*}, \]

\[ E^{*} = (1 + \frac{1}{4} M T^{* -1} T_{p} \nabla, \nabla, \cdots)(\Pi^{*} + \Pi^{*}) \]

\[ - \frac{1}{4} \left[ (T^{* -1/2}) (T_{p} \nabla, \nabla, \cdots) B^{*} \right]^{*}. \]

Two alternative forms for \( E^{*} \) can be deduced from the dimensionless form of Eq. (21),

\[ \Pi^{*} = T^{* 1/2} (v B^{*} + \nabla, B^{*}), \]

leading to the expressions

\[ - E^{*} = \frac{1}{2} T^{* -1/2} (T^{* -1/2} \Pi^{*} - (T^{*} I - T_{p}^{*}) \]

\[ + T_{p}^{*} \nabla B^{*})^{*} + \Pi^{*}, \]  

(37a)

or

\[ - E^{*} = \frac{1}{2} T^{* -1/2} ((T^{*} I - T_{p}^{*}) \nabla B^{*})^{*} + T^{* 1/2} v B^{*} + \Pi^{*}. \]  

(37b)

and

\[ b^{*} = B^{*}, \]  

valid to first order in \( M \). As will be seen later, the last two terms in both expressions for \( E^{*} \) are quadratic in \( v \) and would not enter in a theory linear in \( v \). Conversely, these terms become dominant for large values of the slip velocity \( v \). Notice that although the term \( E^{*} \) has a complicated dependence on \( v \), its dependence on \( T_{p} \) is linear. The second terms in Eqs. (37) are always positive, heating up the gas indefinitely in the direction of motion as soon as \( v \neq 0 \). This effect is counterbalanced by the first term, which tends to bring light and heavy gas temperatures closer. The term \( \Pi^{*} \) is traceless and gives no contribution to the total internal energy, leaving the average heavy temperature \( T_{p} \equiv \text{Trace}(T_{p})/3 \) unaffected.

Here \( \Pi^{*} \) is proportional to \( v^{2} - 3v\nu \), thus tending to extract heat from the direction of motion to put it into the transverse direction. The influence of the different terms will be illustrated in an example (Sec. V.).

III. ASYMPTOTIC FORMS OF THE TRANSFER INTEGRALS

The integrals (31)–(33) can be approximated analytically for the limiting cases \( v \ll 1 \) or \( 1 \ll v \), when the slip velocity \( U_{p} - U \) is much smaller or much larger than the light gas thermal speed \((2kT/m)^{1/2} \). In the first case \( v \ll 1 \), \( \cosh 2\alpha \), \( \sinh 2\alpha \), and \( \exp(-v^{2}) \) can be expanded in a Taylor series about \( v = 0 \), with the following results in terms of the \( \Omega^{*} \) integrals of Hirschfelder et al.\(^{11}\):

\[ \Omega_{m}^{*}(T^{*}) = \frac{2}{m + 1} \int_{0}^{x} x^{2m+3} e^{-x} Q_{m}^{*} (x T^{* 1/2}) dx, \]

(39)

\[ I_{b} = v^{2} \left[ \Omega_{1}^{*} + \frac{3}{2} \Omega_{2}^{*} + \frac{5}{3} \Omega_{3}^{*} \right] v^{4} \cdots \],

(40)

\[ I_{n} = v^{2} \left[ \Omega_{1}^{*} + \frac{3}{2} \Omega_{2}^{*} + \frac{5}{3} \Omega_{3}^{*} \right] v^{4} \cdots \],

(41)

from which \( I_{n} \) follows immediately.

For most values \( v \ll 1 \) the first term gives a very good approximation (Fig. 1). For instance, for \( v = 1, T^{*} = 1 \), the errors in \( I_{b} \) and \( I_{n} \) taking only the first term are, respectively, 2% and 0.4%. For the same temperature, using the three first terms of the expansion, \( I_{b} \) and \( I_{n} \) can be evaluated for \( v \) as large as 2 with errors below 5%.

For large values of \( v \) the integrals (31)–(33) can be rearranged into the form

\[ I = \int_{-\infty}^{\infty} e^{-v^{2}} f_{1}(t, v) dt + \int_{0}^{\infty} e^{-v^{2}} f_{2}(t, v) dt. \]

(42)

A Hermite quadrature gives a good approximation. If \( \{(t_{k}, w_{k})\} \) is the set of abscissas and weight factors for Hermite integration, tabulated in Abramowitz et al.,\(^{12}\) the integral sum (42) can be expressed as

\[ I \approx \sum_{k} f_{1}(t_{k}) w_{k} + \sum_{k} f_{2}(t_{k}) w_{k}. \]

(43)

This series gives a very accurate result for moderate to large \( v \). As shown in Fig. 1, the domains (where the small and large \( v \) approximations are accurate to 5%) do overlap, so that the numerical integration is only required when a greater accuracy is necessary.

For very large \( v \), the Laplace method,\(^{13}\) equivalent to the infinite sum of the Hermite series, yields the following approximations:

![Figure 1: Integrals \( I_{b} \) and \( I_{n} \) as a function of \( v \) for \( T^{*} = 1 \).](image-url)
\[ I_B = \frac{1}{4} \pi^{1/2} v^2 Q_b^*(vT^{*1/2}), \tag{44} \]
\[ I_{B_{\delta}} = \frac{1}{4} \pi^{1/2} v^2 Q_b^*(vT^{*1/2}), \tag{45} \]
\[ I_{B_{\delta}} = v I_B, \tag{46} \]

which are very accurate for \( v > 10 \).

A further simplification can be made by assuming that \( T^{*} \) is very large or very small, which has the physical meaning of making repulsive or attractive forces dominant in the Lennard–Jones potential.

For \( 1 \ll T^{*} \), the molecules behave as point centers of repulsion and the approximation given by Hirschfelder et al.\(^{14}\) for the collision cross sections can be applied:

\[ Q_i = 2\pi (\delta d / (\mu g^2 / 2))^{2/4} A_i(\delta), \tag{47} \]

where \( \delta = 12, \mu = m_p M \), and the coefficients \( A_i \) are given by Hirschfelder et al.\(^{11}\) and \( d = 4e\sigma^{1/2} \). The resulting limiting forms for \( Q_b^* \) and \( \Omega_b^* \) can be substituted in the lower order approximations for \( I_B \) and \( I_{B_{\delta}} \) at \( 1 \ll v \) and \( v \ll 1 \).

For the opposite limit \( T^{*} \ll 1 \), long range attraction prevails over repulsive forces. However, a strictly attractive inverse-sixth potential is inconsistent because low energy collisions would result in molecules merging. The repulsive part has to be retained in a portion of the \( g \) range. But, since the main contribution is a result of the attractive sixth power law, a formula similar to Eq. (47) is to be expected:

\[ Q_j = 2\pi (\delta d/v (\mu g^2 / 2))^{2/4} A_j(\delta), \tag{48} \]

where \( \delta = 6 \) and \( d = 4e\sigma \). This \( A_j(\delta) \) may be estimated from the asymptotic behavior at small values of \( g \) and \( T^{*} \).

A similar expression for \( I_{B_{\delta}} \) follows from Eq. (33). Upon direct substitution in Eqs. (27)–(29), analytical expressions of \( \Pi_b^* \) and \( B_b^* \) are obtained. A plot of \( I_B \) and \( I_{B_{\delta}} \) in the low and high temperature limits is presented in Figs. 2 and 3 where the numerical computation is seen to be practically indistinguishable from the correlations (50) and (51).

The similarity of the terms within parentheses in Eqs. (50a) and (50b) and in Eqs. (51a) and (51b) is not accidental. In effect, using the asymptotic expressions given by Hirschfelder et al.\(^{14}\) for the \( \Omega_b^* \) integrals, a recurrence formula can be obtained:

\[ \frac{\Omega_{b+1}^*}{\Omega_b^*} = \frac{\Gamma(s+3-2/\delta) (s+1)!}{\Gamma(s+2-2/\delta) (s+2)!} = 1 - \frac{2}{\delta (s+2)}, \]

where \( \delta \) is the order of the interaction potential, which according to previous considerations can be taken to be \( 6 \) at low temperatures and \( 12 \) in the opposite limit. This permits us to write expansions (40) and (41) in a simplified fashion for these two limits:

\[ \frac{I_B}{I_{B_{\delta}}} = \frac{1 + \delta \nu^2 - \delta^2 \nu^4 + \cdots}{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}, \quad \delta = 12, \tag{52a} \]
\[ \frac{I_{B_{\delta}}}{I_{B_{\delta}}} = \frac{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}, \quad \delta = 12, \tag{52b} \]
\[ \frac{I_{B_{\delta}}}{I_{B_{\delta}}} = \frac{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}, \quad \delta = 6, \tag{53a} \]
\[ \frac{I_{B_{\delta}}}{I_{B_{\delta}}} = \frac{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}{1 + \frac{1}{\delta} \nu^2 - \frac{1}{\delta^2} \nu^4 + \cdots}, \quad \delta = 6, \tag{53b} \]

where \( I_{B_{\delta}} \) and \( I_{B_{\delta}} \) are the first terms of expansions (40) and (41). Similar expressions can be obtained by binomial ex-
pansion of the fitting equations (50) and (51):

\[
\begin{align*}
I_B &= \begin{cases} 
1 + 0.1532 \nu^2 + \cdots, & 1 \leq T^* \text{ or } 1 \leq \nu T^{1/2} \\
1 + 0.0793 \nu^2 + \cdots, & T^* \ll 1 \text{ and } \nu T^{1/2} \ll 1
\end{cases}, \\
I_{II_1} &= \begin{cases} 
1 + 0.104 \nu^2 + \cdots, & 1 \leq T^* \text{ or } 1 \leq \nu T^{1/2} \\
1 + 0.0537 \nu^2 + \cdots, & T^* \ll 1 \text{ and } \nu T^{1/2} \ll 1
\end{cases}, \\
I_{II_{10}} &= \begin{cases} 
1 + 0.104 \nu^2 + \cdots, & 1 \leq T^* \text{ or } 1 \leq \nu T^{1/2} \\
1 + 0.0537 \nu^2 + \cdots, & T^* \ll 1 \text{ and } \nu T^{1/2} \ll 1
\end{cases}
\]

whose coefficients differ from those of Eqs. (52) and (53) in a range of 9% to 16%. This shows that if the fitting formulas were perfectly accurate in the low \( \nu \) region, the terms within parentheses in Eq. (50a) and (50b) and in Eq. (51a) and (51b) would be exactly coincident. In any case, the similarity between the terms within parentheses observed in Eqs. (50) and (51) permits us to rewrite these expressions in the approximate form:

\[
\begin{align*}
T^* I_B &= \begin{cases} 
0.7573 \nu^{1/3} x, & x \ll 1 \\
1.2467 \nu^{1/6}, & x \ll 1
\end{cases}, \\
T^{1/3} I_{II_1} &= \begin{cases} 
1.040 \nu^{1/6} y, & y \ll 1 \\
1.504 \nu^{1/6}, & y \ll 1
\end{cases}
\]

where \( \nu = \nu T^{1/2} \), \( x = T^* + 0.4678 \nu^2 \), and \( y = T^* + 0.3165 \nu^2 \). The definition of \( x \) and \( y \) is obtained by averaging the parenthesis terms in Eqs. (50) and (51). The former expressions suggest writing \( I_B \) and \( I_{II_1} \) in the following form:

\[
\begin{align*}
T^* I_B &= 1.2467 \nu^{1/6} f(x), \\
T^{1/3} I_{II_1} &= 1.504 \nu^{1/6} g(y)
\end{align*}
\]

where \( f \) and \( g \) are undetermined functions subject to the conditions

\[
\begin{align*}
f &\rightarrow 1 \text{ for } x \rightarrow 0 \quad \text{and} \quad g \rightarrow 1 \text{ for } y \rightarrow 0 \\
f &\rightarrow 0.6074 x^{1/6} \text{ for } x \rightarrow \infty \quad \text{and} \quad g \rightarrow 0.6915 y^{1/6} \text{ for } y \rightarrow \infty
\end{align*}
\]

Numerical computation confirms that indeed, the functions \( f \) and \( g \) are approximately independent of the variable \( \nu \). In Figs. 4 and 5, \( T^* I_B / (1.2467 \nu^{1/6}) \) and \( T^{1/3} I_{II_1} / (1.504 \nu^{1/6}) \) are represented as functions of \( x \) and \( y \), respectively, for different values of \( \nu \). It can be seen that, provided \( \nu \) does not become too large, the curves fall together. Functions \( f \) and \( g \) can be fitted from numerical results by using an interpolation formula of the type

\[
[1 - A (1 - e^{-c^2}) x^a + B x^b]^{1/eb}
\]

In this expression, only \( B \) is given (from the conditions at infinity), so that the four parameters \( A, a, b, \) and \( c \) are unknown. This multiple degree of freedom optimization problem can be easily addressed with the particular expression chosen, because the search for a solution can be done in two stages: first, the fitting function is forced to go through the minimum point of the numerically obtained curves in Figs. 4 and 5. In these points, the factor \( 1 - e^{-c^2} \) can be approximated by unity, so that the relation dependence between \( A, a, b, \) and \( c \) can be explicitly derived, provided that \( b \) is chosen to be the free parameter. Once that \( a \) and \( A \) are known as a function of \( b, b \) is chosen so as to fit the large \( x \) region of the curve, where \( c \) can be ignored. Next, the convenient \( c \), to match with the rest of the curve, is selected.

Our results for \( f \) and \( g \) are

\[
f(x) = [1 - 0.8794(1-e^{-4.25x})x^{0.1271} + 0.0502x]^{1/6},
\]

where \( x = T^* + 0.4678 \nu^2 \) and \( y = T^* + 0.3165 \nu^2 \). The definition of \( x \) and \( y \) is obtained by averaging the parenthesis terms in Eqs. (50) and (51). The former expressions suggest writing \( I_B \) and \( I_{II_1} \) in the following form:

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\[
[1 - A (1 - e^{-c^2}) x^a + B x^b]^{1/eb}
\]
Equations (54) and (55), represented in Figs. 4 and 5, provide a general expression of the transfer integrals for unrestricted $T^*$ and $v$. Notice, however, that the approximation becomes poorer in the vicinity of the minimum point for large $v$; in such cases, Eqs. (44) and (46) can be more convenient. However, in most physical problems, $v$ does not exceed the value $[y/(y - 1)]^{1/2} \sim 1.5$, so that Eqs. (55a) and (55b) are very suitable.

B. Comparison with previous work

A correlation similar to (50b) has been used by Schwartz and Andres for $I_\beta$ in the case $T^* \ll 1, vT^{1/2} \ll 1$. In our notation their expression reads

\[ I_\beta = \frac{0.8848}{2^{3/2}} \left( \frac{1}{2} + 0.3974v^2 \right)^{1/6}, \]

which differs from Eq. (50b) only in the value of the leading coefficient. The difference (40%) is caused by their use of the repulsive inverse-sixth constant $A_v(6)$ to account for the long range attractive interaction constant which has been called $A_v(6)$ in the present work. The low temperature expression (51) for $I_\beta$ can also be compared to Burgess’ calculation of the momentum transfer integral for unrestricted velocity differences, performed under the assumption of hard elastic spheres. This model can roughly describe low-medium temperature range systems. In our notation,

\[ I_\beta = \frac{2v}{3} \left( \frac{3\pi^{1/2}}{8} \left( v + v^{-1} - v^{-3/2} \right) \right) \times \text{erf} \left( \frac{v}{\sqrt{2}} + \frac{v^{-1}}{2} - v^{-3/2} \right), \]

which for the limiting cases $v \ll 1$ and $v \gg 1$, respectively, becomes

\[ I_{\beta_{\ll}} = \frac{2v}{3}, \quad I_{\beta_{\gg}} = \pi^{1/2}v^{3/4}. \]

Our own results are

\[ I_{\beta} = \left( \frac{2v}{3} \right) \Omega_{11}^*(T^*), \quad I_{\beta_{\gg}} = \left( \frac{\pi^{1/2}v^{3/4}}{4} \right) Q_{11}^*(T^{*1/2}). \]

This is consistent with the normalization adopted following Hirschfelder $et$ $al.$, which reduces $\Omega_{11}$ and $Q_{11}$ by dividing them by the corresponding rigid sphere values. Consequently, Eq. (58) can be derived from Eq. (59) by simply setting $\Omega_{11}^* = Q_{11}^* = 1$.

We can compare Burgess’ energy transfer term (scalar) with our resulting energy transfer tensor by setting $T_{\beta}^* = T_{\beta}^*T_{\beta}^*$ and considering the trace of Eq. (37),

\[ -E^{*} = -\text{tr}(E^*)/3 = -\left( T_{\beta}^*/3 \right) \]

\[ \times \left[ (I_\beta/v) + (\partial I_\beta/\partial v)/2 \right] \left( T^* - T_{\beta}^* \right) + vI_\beta T^*. \]

The corresponding expression in Burgess’ equation (15.15) is

\[ -E^{*} = -\left( T_{\beta}^*/3 \right) \left[ \left( \pi^{1/2} \left( v + v^{-1/2} \right) \text{erf} v + v^{-1/2} \right) \times \left( T^* - T_{\beta}^* \right)/2 + vI_\beta T^* \right]. \]

where $I_\beta$ is given by Eq. (57). Low and high velocity limits for Burgess’ expression can again be obtained by setting $\Omega_{11}^*$ and $Q_{11}^*$ equal to 1 in our results. For $v \ll 1$,

\[ -E_{\beta}^* = T^*1/2 \Omega_{11}^* (T^*) (T^* - T_{\beta}^*)/3. \]

For $I_\beta \ll v$, the second term in (60), which represents the dissipation of energy connected with the resistance or drag force caused by the transfer of momentum from the light to the heavy species $[T_{\beta}^*vB_v \Omega_{11}^* in Eq. (37b)]$, prevails over the term proportional to the difference of temperatures:

\[ -E_{\beta}^* = \pi^{1/2}vT_{\beta}^*1/2Q_{11}^*(T_{\beta}^*)^{1/2}/12. \]

As it was pointed out, Eqs. (60)–(63) were obtained assuming $T_{\beta}^* = T_{\beta}^*T_{\beta}^*$. This condition is quite restrictive, since according to our temperature tensor equations, large differences in the parallel and perpendicular components of temperature are to be expected when $1 \ll v$. Therefore, for unrestricted $v$, $T_{\beta}^*$ should be allowed to be a general tensor. The corresponding expression for $E_{\beta}^*$ is obtained by contracting Eq. (37a) and dividing by 3:

\[ -E_{\beta}^* = \left[ (T_{\beta}^* - T_{\beta}^*) + T_{\beta}^*1/2 \right]^2B_v \Omega_{11}^* \]

In the particular limit $v \ll 1$ Eq. (62) remains valid even when the heavy gas is far from equilibrium, provided that we define $T_{\beta}^* = \text{tr}(E_{\beta}^*)/3$. For $v$ of order unity or larger Eqs. (60) and (64) differ considerably because $E_{\beta}^*$ depends separately on the components of the heavy species temperature tensor, $T_{\beta}^*$:

\[ -E_{\beta}^* = (T_{\beta}^*1/2/3) \left[ I_\beta (2T^* - T_{\beta}^*) \right]/(2v) \]

\[ + \Omega_{11} (T_{\beta}^* - T_{\beta}^*) + T_{\beta}^* I_\beta v. \]

For large $v$, the first term in the right-hand side becomes negligible and given that $\Omega_{11} \rightarrow I_\beta$, the same limiting form given in (63) is formally recovered. However, this cancellation of $I_\beta$ and $\Omega_{11}$ is unrealistic because for large $v$, $T_{\beta}^*$ (the component of $T_{\beta}^*$ along the slip direction) will become very large and amplify the small difference $(I_\beta, v - \Omega_{11})$ it makes as important as the other terms. Therefore the expressions involving an average heavy gas temperature can only be used in the low slip velocity limit.

IV. HIGHER-ORDER CORRECTIONS AND RANGE OF VALIDITY

The expansion in powers of $u_{\beta} - U_{\beta}$, which gave rise to the series in (14) and (15) or, in dimensionless form in (34) and (35), becomes poorer as the ratio of the widths of the two distribution functions ceases to be large. Inspection of the dimensionless expansion provides a criterion for the convergence of the series:

\[ T_{\beta}^* \ll T^*/M. \]

For large values of $v$, the range of convergence is even broader, as can be seen from the order of magnitude of $P_{\beta}^\perp \nabla \nabla \cdot $,

\[ T_{\beta}^* \ll v^2 T^*/M. \]

As $T_{\beta}^*$ grows closer to the limit value given in (66) and (67), the importance of the next-order correction in $M$ becomes greater. This additional term can easily be introduced by carrying out the direct differentiation indicated in Eqs. (34) and (35), which is straightforward given the simple form of correlations (50). In this way, analytic expressions for the
high and low temperature ranges are obtained; they will not be included here because of their complexity. Notice however that a systematic theory including first-order corrections in $M$ must incorporate our modification in the definition of $T^*$ [Eq. (23)]. It also has to account for the departure of the distribution function $f$ from a Maxwellian resulting from the nonhomogeneity of the velocity and temperature fields.

Under the assumption of number density dilution, $n_p \ll n$, the kinetic influence of the heavy gas on the light gas distribution function $f$ can be neglected, so that $f$ can be described in terms of the pure gas Chapman–Enskog solution. If we also approximate the solutions of the integral equations appearing in the theory by the first term in their Sonine polynomial expansion, we may write

$$f(u) = n(2\pi k T/m)^{-3/2}e^{-e^2/2} \left[ 1 - \mu c(e^2 - 1) - K e c \right],$$

(68)

where $c = \sqrt{(u-U)/(2k T/m)^{1/2}}$, $e$ is the dimensionless temperature gradient,

$$e = (3\mu/2p)(2k T/m)^{1/2} \nabla \ln T,$$

(69)

and $K$ is the dimensionless, symmetrized, and traceless velocity gradient tensor,

$$K = \mu [\nabla(U-\nabla U)-2/3(\nabla \cdot U)].$$

(70)

Using the expression for the cross collision operator given by Fernandez de la Mora and Mercer$^7$ for the limit of small $u$, we obtain the following expressions, valid to first order in $M$ and $v$:

$$B^* = \left( T^* / 3 \right) \left[ 2v \cdot \left[ \Omega_{11}^* \mathbf{I} - \left( \frac{3}{2} \Omega_{11}^* - \frac{1}{2} \Omega_{11}^* \right) \mathbf{K} \right] \right] + e \left[ \Omega_{11}^* \mathbf{I} - \left( \frac{3}{2} \Omega_{11}^* - \frac{1}{2} \Omega_{11}^* \right) \mathbf{K} \right],$$

(71)

$$E^* = \left( T^* / 3 \right) \left[ \Omega_{11}^* \mathbf{I} - \frac{1}{2} \Omega_{11}^* \mathbf{K} \right],$$

(72)

while the full nonlinear $v$ dependence of $B^*$ and $E^*$ on the new $K$ and $e$ terms could be evaluated similarly as done before. We note that when the absence of velocity slip there is an acceleration $B^*$ leading to the drift of heavy molecules down temperature gradients caused by the term proportional to $e$ in Eq. (71). This thermal force degenerates into the thermal diffusion effect under near equilibrium conditions. In these conditions, the $K$ term does not contribute to the net force, but it makes a contribution to the energy transfer term $E^*$. The order of magnitude of this contribution is only a fraction $M$ of that of the analogous compression heating terms $P_p(\nabla U_p)$, $(P_p \nabla U_p)$ in the left-hand side of Eq. (3).

V. HEAVY GAS PénéTRATION INTO A LIGHT GAS AT REST

In this section, the expressions obtained previously will be applied to the study of heavy molecules penetrating into a stagnant light gas. The impingement of fast heavy molecules on a target gas at rest is an approximate model for different phenomena occurring in gas mixture flows. For instance, the shock layer that develops in front of a probe or any other obstacle immersed in a supersonic field can roughly be considered a light gas stagnation zone through which the heavy molecules keep a large fraction of their former speed, while their temperatures may overshoot significantly above the gas temperature.

The starting point is Eqs. (1)–(3), where two assumptions are introduced: first, the flow is assumed to be unidirectional; along the $z$ direction; and, second, the pressure tensor is neglected in the momentum equation and the heat flux in the energy equation. For $P$ and $Q$ to be negligible, respectively, in Eqs. (2) and (3), it is necessary that

$$k T_p \ll m_p U_p^2,$$

(73)

which turns out to be a hypersonic condition stating that the Mach number of the heavy species has to be much larger than unity. For simplicity's sake the constraint $\rho \ll \rho_p$ (small heavy gas mass loading) is added, so that the self-collision energy tensor transfer term $[E_{self}$ in Eq. (3)] can be neglected.

With these restrictions Eqs. (1)–(3) become

$$U_p U_{pz} = b,$$

(74)

$$(k/m_p) U_p T_{irz} = E_{ir} - 2T_{ir} k T_{pz} / m_p,$$

(75)

$$(k/m_p) U_p T_{ipz} = E_i,$$

(76)

where $b = \|b\|$, and $T_{ir}$, $E_{ir}$ and $T_{ip}$, $E_i$, are, respectively, the components along the $z$ direction and in any perpendicular direction of the temperature and energy transfer terms. Equations (74)–(76) can be normalized by defining

$$v = U / U_0, \quad t = t_0 / K, \quad x = z / L_0,$$

(77)

where $U_0 = (2e T^* / \mu)^{1/2}$, so that the definition of $v$ agrees with that given in Eq. (25), $t_0 = U_0 / K_B$, and $L_0 = t_0 U_0$. The variable $\tau$ will be used later. Notice how the deceleration scale is determined by the interaction potential. Denoting $d / dx = ' \cdot$, Eqs. (74)–(76) become

$$v' = B^*,$$

(78)

$$v T_{ipz} + 2 T_{ipz} v' = -4 T^* - 1 / 2 E_{ip},$$

(79)

$$v T_{irz} = -4 T^* - 1 / 2 E_{ir},$$

(80)

where $B^*$ is given by (29) and

$$- E_{ip} = [I_{ir} (T^* - T_{ir}) / v T_{ipz} + I_{ir} I_{ir} / 3] T^* 1 / 2,$$

(81)

$$- E_{ir} = [I_{ir} / 2 (v (T^* - T_{ipz}) + I_{ip} / 6) T^* 1 / 2].$$

(82)

This system is to be solved given initial conditions for the impinging heavy gas:

$$v = v_{ip}, \quad T_{ip} = T_{ip0}, \quad T_{ir} = T_{ir0}, \quad \text{at} \; x = 0.$$

A. Limit $v_0 \ll 1$

If $v_0$ is small enough, the approximations (40) and (41) hold and we are left with

$$v' = -2 \Omega_{11}^* T^* / 3,$$

(83)

$$T_{ipz} = 4 T^* \Omega_{11}^* / (3 v),$$

(84)

$$T_{ipz} = 4 T^* \Omega_{11}^* (T^* - T_{ip}) / (3 v),$$

(85)

which can easily be solved to yield

$$V = 1 - x / (S' v_0),$$

(86)

$$T_{ipz} = T_{ipz0} - 2 T_{ip} * V,$$

(87)

$$T_{ipz} = T^* + (T_{ipz0} - T^*) V^2.$$

(88)
B. Limit 1 $\ll v_0$

For large $v_0$ a new limit for Eqs. (77)–(79) is reached. Substituting expressions (44)–(46), the following set of equations is obtained:

$$v' = - T^* Q' \frac{\pi^{1/2} v}{4},$$

$$v T^*_{ip} + 2 T^*_{ip} v' = T^* Q^* \frac{\pi^{1/2} v^3}{3},$$

$$v T^*_{ip} = \left[ 2 \left( T^* - T^*_0 \right) Q^* - 2 T^* Q^* v^3 \right] \frac{\pi^{1/2} T^*}{4}.$$

Provided that $v^{-2} \ll T^*$, which is a condition very easily met because $1 \ll v_0$, $Q^*$ and $Q^*_0$ can be written as

$$Q^* = 1.319 v^{-1/3} T^* x^{-1/6},$$

$$Q^*_0 = 1.210 Q^*.$$

In this limit, the following solution is obtained:

$$V = \left( 1 - x^* / 3 \right)^{\frac{3}{2}},$$

$$\theta_{ip} = \frac{C(1 - V^4) / \sqrt{V^2}},$$

$$\theta_{ip} - \theta_{ip0} = - D V^2 \ln V,$$

where

$$V = v / v_0, \quad \theta_{ip} = T_{ip} / (T_{ip0} v_0^2),$$

$$x^* = \frac{1}{2} Q^* T_{ip} x, \quad Q^*_0 = Q^* (T^*, v_0),$$

$$C = 0.596 76, \quad D = 0.806 47.$$

These expressions are restricted again by the condition $\theta_{ip} \ll V^2 / M$. Figure 7 shows the predicted shape of $V$, $\theta_{ip}$, and $\theta_{ip}$ as a function of $x^*$, together with the results for numerically computed transfer terms: as can be seen, $V$ and $\theta_{ip}$ are accurately predicted by formulas (95) and (97), but Eq. (96) only agrees with the computed $\theta_{ip}$ in the first stages of the deceleration. The same discussion made about Eq. (65) applies here: The otherwise negligible difference $I_{ip} - v_{ip} = (\partial I_{ip} / \partial v)/2$ is amplified through the factor $T_{ip}$ becoming as important as the other terms. This causes $T^*_{ip}$ to reach a peak and then decrease.

C. Stopping distance

The hypersonic assumption decouples the momentum from the energy equation, so that the problem can be independently solved for the velocity in a simple way. A single curve giving the stopping distance $x^*_s = x_0 / S'$ as a function of the initial velocity is obtained for each temperature (Fig. 7).
FIG. 8. Stopping distance $x_0^*$ as a function of initial velocity $v_0$. Figure 8(b) is an expanded Fig. 8(a) for moderate values of $v_0$: --- High temperature limit, obtained by integration of Eq. (50a); --- low $v_0$ limit; --- --- $T^* = 0.1$; --- --- $T^* = 1$; --- --- $T^* = 10$.

FIG. 9. Penetration depth as a function of time for different values of $v_0$ and $T^*$. Anderson's results using Monte Carlo simulation for $M = 100$ and inverse-12th repulsive potential.

expression obtained by averaging the velocity change of the impinging molecules as they collide with target molecules at rest. The effect of the target molecules' motion is estimated by using an approximate expression for the collision rate. His results are thus more accurate when the stagnant gas molecules have a thermal speed much smaller than the mean velocity of the incident molecules, that is, $1 \ll v_0$. His results for the 12th potential can be written as

$$\frac{d\nu}{dT} = -0.471 \nu^{5/3} T^{5/6} \left( 1 + \frac{4}{\pi \nu^2} \right)^{1/2},$$

while our expression for $1 \ll T^*$ [Eqs. (50) and (74)] can be rewritten as

$$\frac{d\nu}{dT} = -0.584 \nu^{5/3} T^{5/6} \left( 1 + \frac{2.715}{\nu^2} \right)^{1/3}.$$  

A difference of about 20% is observed. The terms within the parentheses, when expanded in $v^{-1}$, respectively become $(1 + 0.6366/v^2 + \cdots)$ and $(1 + 0.7252/v^2 + \cdots)$. Differences probably arise from the approximate expression for the collision rate used by Anderson.

VI. CONCLUSIONS AND LIMITATIONS OF RESULTS

In this paper, simple formulas are given for the cross-collision integrals describing the light-heavy momentum and energy (tensor) transfer in a binary disparate-mass gas mixture. Our results, to lowest order in $m/m_p$, are subject to the constraint that the molar fraction of the heavy gas be moderately small ($n_p/n \leq 1$) so that the distribution function of the light gas can be considered unaffected by the presence of the heavy molecules. However, this is not a severe restriction in disparate-mass mixtures, since large heavy gas mass fractions are compatible with $n_p \leq n$. Furthermore, the complementary situation can easily be handled by using an equilibrium two-fluid approach, which describes both species distribution functions as Maxвелlians. Only to go one step further in our perturbation technique in powers of $m/m_p$ is one forced to impose the more restrictive condition $n_p \ll n$ so that $f$ can be described in terms of the first-order pure gas Chapman-Enskog solution. Our equations are also subject to the restriction that the relation $S$ between the interspecies relaxation time and the characteristic macroscopic time be at most of order unity. This constraint precludes
extreme nonequilibrium conditions, in which the light gas distribution also departs considerably from the Maxwellian distribution.

On the other hand, this work is only concerned with the cross-collision integral terms, $b$ and $E$ in Eqs. (2) and (3). Given that the self-collision integral terms become important as soon as the mass fractions are comparable ($\rho \sim \rho_p$), some model for the heavy gas distribution functions is required in order to compute the self-collision integral tensorial energy transfer term $E_{self}$. In Sec. V the heavy gas temperature tensor was determined without a model for $E_{self}$ thanks to the assumption $\rho \ll \rho_p$.

The expressions for the interspecies transfer are based on the disparity of thermal speeds between the two species, which requires $T_p$ to be much smaller than $m_p T/m$. Under extreme circumstances (i.e., a hot beam encountering a cool light gas) this assumption is not fulfilled, but in practically all relevant situations $T_p$ and $T$ are of the same order, thus ensuring the legitimacy of our expansion. This condition should not be confused with the hypersonic requirement which was added for simplicity in the example of Sec. V.

Finally, the normalization and the resulting correlations are based on the Lennard–Jones interaction model, which provides a simple and realistic representation for spherical nonpolar molecules at a wide range of temperatures. The extension of these results to other interaction potentials is straightforward.

ACKNOWLEDGMENTS

The present work has been supported by a cooperative research grant from Schmitt Technologies Associates and the State of Connecticut (number 885-176). P. Riesco-Chueca is sponsored by the U.S.–Spanish Joint Committee for Cultural and Educational Cooperation.

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