

Nonparallel linear stability analysis of Long's vortex

R. Fernandez-Feria

Universidad de Málaga, E.T.S. Ingenieros Industriales, 29013 Málaga, Spain

(Received 27 March 1998; accepted 5 January 1999)

A nonparallel linear stability analysis of a family of self-similar vortex cores which includes Long's vortex as a particular member is performed using parabolized stability equations (PSE). The resulting streamwise variation of both the spatial growth rate and the axial wave number of the different unstable modes is compared with the results from a local spatial stability analysis which also takes into account the effects of viscosity and of the streamwise variation of the basic flow, so that the effect of the history of the disturbances on their stability is quantified. It is shown that this last effect is negligible for high Reynolds numbers, but becomes increasingly important as the Reynolds number decreases, especially for very small growth rates. The marching method used to solve the PSE is computationally much faster than the standard methods for solving the nonlinear eigenvalue problem resulting from the local stability equations. As a new result, the local spatial calculations reveal the existence of unstable counter-rotating spiral modes with negative group velocities for Type II Long's vortices (that is, vortices with negative streamwise velocity at the axis), thus showing that these flows are subcritical in Benjamin's sense. This kind of instability does not appear for Type I vortices, which can only sustain non-axisymmetric convective instabilities, and are therefore supercritical. Thus, the spatial stability analysis establishes a fundamental distinction between Type I and Type II Long's vortices. © 1999 American Institute of Physics. [S1070-6631(99)00205-6]

I. INTRODUCTION

Long's vortex has been extensively considered as a simple model for high Reynolds number vortices of geophysical and engineering interest, mainly because it is an exact solution to the near-axis approximation of the Navier–Stokes equation which is nonparallel and consistently includes a relatively important axial flow, both characteristics present in most real vortices of interest. In particular, its stability has been analyzed by a number of authors using different techniques and degrees of approximation, with the main objective of trying to elucidate and predict some of the interesting properties that highly swirling flows present in practice. Most of these previous works considered the temporal stability (that is, with given real wave number and unknown complex frequency) of Long's vortex using a parallel flow approximation. Thus, Foster and Duck¹ analyzed the inviscid stability using a finite-difference method to solve the resulting set of two ordinary differential equations. They found that the flow is unstable for non-axisymmetric, counter-rotating ($n < 0$) perturbations. This work was later extended by Foster and Smith,² and by Ardalan *et al.*,³ who made an asymptotic analysis of the inviscid stability in the limit of large flow force for Type II and Type I Long's vortices, respectively. The viscous stability was considered by Khorrami and Trivedi,⁴ who, using a spectral collocation method to solve the stability equations, obtained also asymmetric co-rotating ($n > 0$) inviscid unstable modes for Type II vortices, previously found asymptotically in the limit of large flow force by Foster and Smith.² Nonparallel effects were considered locally by Foster and Jacqmin⁵ in the limit of large flow force and small axial wave number for Type II Long's vortices. They found that for these long-wave modes

the nonparallelism of the flow is more important than viscosity in the finite Reynolds behavior of the perturbations. The local nonparallelism of the basic flow and of the amplitude of the perturbations, in addition to the effect of viscosity, were taken into account in general in Ref. 6, where new purely viscous and inviscid axisymmetric, unstable modes were found for Type II Long's vortices. These last unstable modes have growth rates more than an order of magnitude smaller than the inviscid non-axisymmetric unstable modes.

The last two cited works considered the nonparallel effects of the basic flow *locally*, i.e., without the effect of the history of the perturbation, as it is convected by the basic flow, upon its stability. The new terms accounting for this effect in the stability equations are, for high Reynolds numbers, of the same order of magnitude than the terms arising from the nonparallelism of the basic flow and from the viscous forces (when the axial wavelength of the disturbances is comparable to the vortex core radius). Hence, they should be taken into account in a consistent analysis of the convective stability of Long's vortex. However, we shall see that the actual effect of these new terms is negligible except for relatively small Reynolds (Re) numbers when the growth rate is very small.

To appropriately account for the effect of the history of the disturbance on its stability, one has to consider the *spatial*, rather than the *temporal*, stability of the flow; that is, with given real frequency look for the complex axial wave number. This kind of analysis is also the appropriate one to study the spatial evolution of waves as they propagate from a given forced oscillation at a given location. In order to check the numerical procedures and results, a comparison is first made between the results of the *local* spatial stability equa-

tions (without the terms accounting for the history of the disturbances) and the equivalent results from the *temporal* stability analysis of Ref. 6 (hereafter T) using Gaster's relation⁷ for small growth rates.⁸ It is shown that the local spatial analysis reproduces the results of the temporal instability calculations when the group velocity is positive (convective instabilities). However, new unstable counter-rotating spiral modes with negative group velocities are found for Type II vortices with the present spatial formulation which are not obtained with the temporal analysis. As discussed in the last section, these new unstable modes establish a fundamental difference between Type I and Type II Long's vortices. Once the results from the spatial and temporal local eigenvalue problems have been checked with each other, they are compared with the nonlocal spatial results of the linear stability equations with the additional terms describing the history of the disturbances. Retaining terms up to order Re^{-1} , these stability equations are equivalent to those obtained with the assumptions made in the *parabolized stability equations* (PSE) method, formulated by Bertolotti, Herbert, and Spalart⁹ for the Blasius boundary layer problem (see also Refs. 10 and 11). A marching technique in the streamwise direction, combined with a spectral collocation method in the radial direction, is used to solve the resulting PSE for the stability of Long's vortex.

II. FORMULATION OF THE PROBLEM

A. The basic vortex

Long's vortex^{12,13} is a similarity solution to the near-axis boundary layer approximation of the steady, incompressible, and axisymmetric Navier–Stokes equations, matching an inviscid flow with axial and azimuthal velocities inversely proportional to the distance r to the axis of symmetry. Long showed that there are two solutions for $M > M^*$, and none for $M < M^*$, where M is the dimensionless flow force, and M^* is a critical value. It was shown in Ref. 14 that Long's vortex can be viewed as a particular member of a more general family of self-similar vortex cores matching an inviscid flow field proportional to a certain power of the distance to the axis of symmetry; that is, vortices which in cylindrical polar co-ordinates (r, θ, z) match an inviscid velocity (U, V, W) and pressure P fields given by

$$\begin{aligned} W &= W_0 r^{m-2}, \quad U = 0, \quad V = \pm L W_0 r^{m-2}, \\ \frac{P}{\rho} &= \frac{(L W_0)^2}{2(m-2)} r^{2(m-2)}, \end{aligned} \quad (1)$$

where ρ is the fluid density, $0 < m < 2$, and W_0 and the swirl parameter L are positive constants (U is not exactly zero, but decays faster than W or V for $r \rightarrow \infty$).¹⁴ Long's vortex corresponds to the case $m = 1$. It was also argued in Ref. 14 that vortices with m slightly larger than unity are the most interesting ones within the family to model real vortex flows. However, since it was shown in T that the temporal stability properties are qualitatively the same for $m = 1$ and for $m > 1$ if one uses the nondimensional axial velocity at the axis (parameter A_1 defined below) instead of M or L as the parameter characterizing the different solutions, only Long's

vortex ($m = 1$) shall be considered in this work, keeping in mind that the results can be extrapolated to the more interesting cases with m slightly larger than unity. In fact, for the sake of generality, the problem will be formulated for any value of m , although results will be given only for Long's vortex.

The vortex core has the self-similar structure

$$\Psi = \nu z f(\xi), \quad (2)$$

$$V = \frac{\nu z}{\delta^2(z)} \gamma(\xi), \quad \frac{P}{\rho} = \frac{(\nu z)^2}{\delta^4(z)} \beta(\xi), \quad (3)$$

where Ψ is the stream function for the meridional motion, through which the axial and radial velocity components are

$$\begin{aligned} W &= \frac{1}{r} \frac{\partial \Psi}{\partial r} = \frac{\nu z}{\delta^2(z)} 2 f'(\xi), \\ U &= -\frac{1}{r} \frac{\partial \Psi}{\partial z} = -\frac{\nu}{r} \left[f(\xi) - \frac{2 \xi f'(\xi)}{m} \right]. \end{aligned} \quad (4)$$

The similarity variable ξ is defined in terms of the vortex core thickness $\delta(z)$:

$$\xi = \eta^2, \quad \eta = \frac{r}{\delta(z)}, \quad \delta(z) = \left(\frac{m \nu z}{W_0} \right)^{1/m}. \quad (5)$$

The functions f , γ , and β are governed by a set of three nonlinear ordinary differential equations, which are solved in Ref. 14 by shooting. For each m , the swirl parameter L is a function of the nondimensional axial velocity at the axis, $A_1 \equiv f'(0)$, with $-1/\sqrt{2} < A_1 < \infty$. Except for the case $m = 1$, this function is nonmonotonic, having an extremum $L^*(m)$, which is a minimum for $0 < m < 1$ and a maximum for $1 < m < 2$ (see figures 2 and 3 in Ref. 14). Thus, when $0 < m < 1$, two solutions exist for $L > L^*(m)$, and there is no solution for $L < L^*(m)$; when $1 < m < 2$, no solutions exist for $L > L^*(m)$, and there are two possible solutions for $L < L^*(m)$. For the special case $m = 1$ (Long's vortex), $L = \sqrt{2}$ for all values of A_1 . In this case, the nondimensional flow force parameter M , which is constant along the axis for $m = 1$ (see the Appendix for a discussion of this matter), plays a role somewhat analogous to L for $m \neq 1$:^{13,15} for $M > M^* \approx 3.75$ two solutions exist, denoted by Burggraf and Foster¹⁵ as Type I and Type II solutions, and none for $M < M^*$. For $m > 1$, this classification of the solutions is extrapolated to the two possible solutions for $L < L^*(m)$. In terms of the nondimensional axial velocity at the axis, A_1 , there is no distinction between $m = 1$ and $m > 1$: All Type I solutions have a positive axial velocity at the axis, with A_1 within the interval $(A_1^*(m), \infty)$, where $A_1^* \approx 0.15$ is almost independent of m ; most Type II solutions have a negative axial velocity at the axis, with A_1 in the interval $(-1/\sqrt{2}, A_1^*(m))$ (see figures 1 and 2 in T). In the limit of large A_1 (M large for $m = 1$ or $L \rightarrow 0$ for $m > 1$), Type I vortex has an intense positive axial flow near the axis, while for $A_1 \rightarrow -1/\sqrt{2}$ (again M large for $m = 1$ or $L \rightarrow 0$ for $m > 1$), Type II vortex has the form of a ring-jet with large positive axial flow on the ring and negative axial flow in its interior (see Ref. 2 for asymptotic solutions in these two limits when $m = 1$). Finally, it should be mentioned here that

although Long's vortex ($m = 1$) is the only one in the family of self-similar solutions which has a constant, nonzero flow force M , all the other solutions for $0 < m < 2$ satisfy the weaker condition (see Appendix)

$$\lim_{K \rightarrow \infty} \int_0^{K\delta(z)} \frac{\partial}{\partial z} \left(W^2 + \frac{P}{\rho} \right) r dr \rightarrow 0. \quad (6)$$

B. Nonparallel linear stability formulation

The nonparallel parabolized stability equations are now developed for the above family of swirling jets. The flow variables, (u, v, w) and p , are decomposed, as usual, into a mean part, (U, V, W) and P , and a small perturbation. After (3) and (4),

$$w = \frac{\nu z}{\delta^2} [2f' + \bar{w}], \quad u = \frac{\nu}{r} \left[-f + \frac{2\xi f'}{m} + \frac{rz}{\delta^2} \bar{u} \right], \quad (7)$$

$$v = \frac{\nu z}{\delta^2} [\gamma + \bar{v}], \quad \frac{p}{\rho} = \frac{(\nu z)^2}{\delta^4} [\beta + \bar{p}], \quad (8)$$

where the perturbations

$$\mathbf{s} = [\bar{u}, \bar{v}, \bar{w}, \bar{p}]^T \quad (9)$$

are, in general, functions of the four independent variables (r, θ, z, t) . Since the mean flow depends on the similarity variable ξ [Eq. (5)], this nondimensional variable is used instead of r . A nondimensional axial coordinate x is also defined:

$$x = \frac{z}{z_0}, \quad (10)$$

where z_0 is a characteristic (or initial) axial distance. The use of this axial scale length in addition to the radial characteristic length δ_0 , defined as the vortex thickness at z_0 , $\delta_0 = (m\nu z_0/W_0)^{1/m}$, allows the definition of the small parameter

$$\Delta_0 = \frac{\delta_0}{z_0} \ll 1, \quad (11)$$

in terms of which one can derive the PSE from the complete stability equations in a more consistent way than the procedure of just neglecting second- and higher-order streamwise derivatives.

The perturbations (9) are decomposed in the standard form:⁹

$$\mathbf{s}(x, \xi, \theta, t) = \mathbf{S}(x, \xi) \chi(x, \theta, t), \quad (12)$$

where, as the main difference with the temporal analysis in T, the complex eigenfunctions

$$\mathbf{S}(x, \xi) = [iF(x, \xi), G(x, \xi), H(x, \xi), \Pi(x, \xi)]^T, \quad (13)$$

are allowed to depend on the axial coordinate x independently to the radial one. The other part of the perturbation is an exponential that describes the wave-like nature of the disturbance,

$$\chi(x, \theta, t) = \exp \left[\frac{1}{\Delta_0} \int_{x_0}^x a(x') dx' + i(n\theta - \Omega t) \right]. \quad (14)$$

The nondimensional, order of unity (or smaller), axial wave number a is defined as

$$a(x) \equiv \delta_0 k(x) \equiv \gamma(x) + i\alpha(x), \quad (15)$$

which accounts for the fast, wave-like variation of the perturbations. Its real part $\gamma(x)$ is the exponential growth rate, and the imaginary part $\alpha(x)$ is the axial wave number. A nondimensional, order of unity, frequency ω is also defined:

$$\omega \equiv \frac{\Omega \delta_0^3}{\nu z_0}. \quad (16)$$

Substituting (12)–(16) into the incompressible Navier–Stokes equations, and neglecting second-order terms in both the small perturbations and the inverse of the local Reynolds number,

$$\Delta \equiv \frac{\delta(z)}{z} = \Delta_0 x^{1/m-1} \equiv \text{Re}^{-1}, \quad (17)$$

which is assumed to be small within the boundary layer approximation (note that for Long's vortex $\Delta = \Delta_0$ is constant along the flow, while for m slightly larger than unity Δ is a slowly decreasing function of x), the following set of linear PSE results:

continuity:

$$-x\Delta \frac{\partial H}{\partial x} = 2i \frac{\partial(\xi^{1/2} F)}{\partial \xi} + in \frac{G}{\xi^{1/2}} + a^* H + \Delta \left[\frac{m-2}{m} H - \frac{2\xi}{m} \frac{\partial H}{\partial \xi} \right]; \quad (18)$$

r -momentum:

$$\begin{aligned} -2f'x\Delta \frac{\partial F}{\partial x} &= -4\Delta\xi \frac{\partial^2 F}{\partial \xi^2} - 2\Delta(f+2) \frac{\partial F}{\partial \xi} + \left[-i\omega^* + \frac{in\gamma}{\xi^{1/2}} + 2f'a^* \right. \\ &\quad \left. + \Delta \left(\frac{1+f+n^2}{\xi} - \frac{f'}{m} + \frac{4\xi f''}{m} - (a^*)^2 \right) \right] F \\ &\quad + \left(\frac{2i\gamma}{\xi^{1/2}} + \Delta \frac{2n}{\xi} \right) G - 2i\xi^{1/2} \frac{\partial \Pi}{\partial \xi}; \end{aligned} \quad (19)$$

θ -momentum:

$$\begin{aligned} -2f'x\Delta \frac{\partial G}{\partial x} &= -4\Delta\xi \frac{\partial^2 G}{\partial \xi^2} - 2\Delta(f+2) \frac{\partial G}{\partial \xi} + \left[-i\omega^* + \frac{in\gamma}{\xi^{1/2}} + 2f'a^* \right. \\ &\quad \left. + \Delta \left(\frac{2(m-1)}{m} f' + \frac{n^2+1-f}{\xi} - (a^*)^2 \right) \right] G \\ &\quad + \left[2i(\xi^{1/2}\gamma)' + \Delta \frac{2n}{\xi} \right] F + \Delta \left(\frac{m-2}{m} \gamma - \frac{2\xi}{m} \gamma' \right) H \\ &\quad + \frac{in}{\xi^{1/2}} \Pi; \end{aligned} \quad (20)$$

z-momentum:

$$\begin{aligned}
 & -2f'x\Delta\frac{\partial H}{\partial x} - x\Delta\frac{\partial \Pi}{\partial x} \\
 & = -4\Delta\xi\frac{\partial^2 H}{\partial \xi^2} - 2\Delta(f+2)\frac{\partial H}{\partial \xi} + \left[-i\omega^* + \frac{in\gamma}{\xi^{1/2}} + 2f'a^* \right. \\
 & \quad \left. + \Delta\left(4\frac{(m-2)}{m}f' - \frac{4\xi}{m}f'' + \frac{n^2}{\xi} - (a^*)^2\right)\right]H \\
 & \quad + 4i\xi^{1/2}f''F + \left[a^* + \Delta\frac{2(m-2)}{m}\right]\Pi - \Delta\frac{2\xi}{m}\frac{\partial \Pi}{\partial \xi}; \quad (21)
 \end{aligned}$$

where

$$a^* \equiv ax^{1/m}, \quad \omega^* \equiv \omega x^{3/m-1}. \quad (22)$$

The above approximation, where terms $O(\Delta^2)$ and smaller are neglected, is consistent with the boundary layer approximation of the basic flow, where terms $O(\Delta^2)$ are also neglected.¹⁴ The retained $O(\Delta)$ terms account for three different effects on the stability of the perturbations: (i) the effect of viscosity, (ii) the effect of the nonparallelism of the basic flow and of the amplitude of the perturbations, and (iii) the effect of the history, or convective evolution, of the perturbations. All these three effects are therefore negligible in the limit of very high Reynolds numbers ($\Delta \rightarrow 0$). The last mentioned effect (iii) is described by the streamwise derivatives on the left-hand side of the stability equations (18)–(21). These terms are therefore the responsible ones for the partial differential (though parabolic) character of the equations. They are the only terms in (18)–(21) ignored in the linear stability analysis of T, because they cannot be appropriately included in a temporal stability analysis, which has to be local: one fixes a real axial wave number α ($a^* = i\alpha$) at a given axial location x (which is embedded in a^* and ξ), and looks for the complex frequency ω^* . Here, a spatial stability analysis including the effect of the history of the disturbances will be performed by solving the above parabolized equations by a marching technique. The integration is started at a given axial location x_0 with a fixed real frequency ω , to find the corresponding complex axial wave numbers $a(x)$. For given flow parameters and azimuthal wave number n , an unstable mode appears at a location x if the real part of $a(x)$ becomes positive.

As it stands, there is some ambiguity in the partition of the perturbations (12) into two functions of x . To close the problem one has to enforce an additional condition which puts some restriction on the axial variation of $\mathbf{S}$.^{9,10} Basically, one uses a normalization condition that restricts rapid changes in x of $\mathbf{S}$, in accordance with the slow axial variation of the basic flow. Thus, the growth rate and the axial sinusoidal variation is represented by the exponential function χ . Two such normalization conditions will be used here, both based on a physical growth rate and a wave number defined in terms of the axial velocity perturbation $\tilde{w} \equiv (\nu z / \delta^2) \bar{w} = (\nu z / \delta^2) H \chi$. The first one makes use of the physical

growth rate defined at a particular radial distance ξ_0 , e.g., the point where the absolute value $|H(x, \xi)|$ reaches a maximum for each axial location x :

$$\begin{aligned}
 \bar{a}_1(x) & \equiv \bar{\gamma}_1(x) + i\bar{\alpha}_1(x) \\
 & \equiv \left(\frac{\delta_0}{\tilde{w}} \frac{\partial \tilde{w}}{\partial z} \right)_{\xi=\xi_0} = a(x) + \frac{m-2}{m} \frac{\Delta_0}{x} + \frac{\Delta_0}{H(x, \xi_0)} \\
 & \quad \times \left[\frac{\partial H(x, \xi_0)}{\partial x} - \frac{2\xi_0}{mx} \frac{\partial H(x, \xi_0)}{\partial \xi} \right]. \quad (23)
 \end{aligned}$$

The other one is based on the physical growth rate defined in terms of the radial integral of the axial velocity perturbation:

$$\begin{aligned}
 \bar{a}_2(x) & \equiv \bar{\gamma}_2(x) + i\bar{\alpha}_2(x) \\
 & \equiv \delta_0 \frac{\int_0^\infty \tilde{w}^\dagger (\partial \tilde{w} / \partial z) dr}{\int_0^\infty |\tilde{w}|^2 dr} \\
 & = a(x) + \frac{m-2}{m} \frac{\Delta_0}{x} + \Delta_0 \\
 & \quad \times \frac{\int_0^\infty (H^\dagger / \xi^{1/2}) (\partial H / \partial x) d\xi}{\int_0^\infty (|H|^2 / \xi^{1/2}) d\xi} \\
 & \quad - \frac{2\Delta_0}{mx} \frac{\int_0^\infty \xi^{1/2} H^\dagger (\partial H / \partial \xi) d\xi}{\int_0^\infty (|H|^2 / \xi^{1/2}) d\xi}, \quad (24)
 \end{aligned}$$

where $\dagger$ denotes the complex conjugate. In terms of these two complex, physically defined, wave numbers $\bar{a}_1$ and $\bar{a}_2$, the two different normalization conditions used here can be expressed as $\bar{a}_1(x) = a(x)$ and $\bar{a}_2(x) = a(x)$, respectively, for all $x \geq x_0$. That is, at each axial step, all the terms on the right-hand side of (23) or (24), except for the first one, are set equal to zero, thus transferring the main part of the streamwise variation of the perturbations to the exponential function χ . We shall see that both normalization conditions yield almost identical results, except when $|H|$ have several maxima, in which case the condition based on $\bar{a}_2$ is preferred.

The PSE with its normalization condition is solved with the radial boundary conditions:¹⁶

$\xi \rightarrow \infty$:

$$F = G = H = 0; \quad (25)$$

$\xi = 0$:

$$F = G = 0, \quad \partial H / \partial \xi = 0 \quad (n=0), \quad (26)$$

$$F \pm G = 0, \quad \partial F / \partial \xi = 0, \quad H = 0 \quad (n = \pm 1), \quad (27)$$

$$F = G = H = 0 \quad (|n| > 1). \quad (28)$$

In addition, the eigenfunction $\mathbf{S} = [iF, G, H, \Pi]^T$ and the complex wave number a have to be specified at the initial axial location x_0 (see below).

C. Numerical method

Equations (18)–(21) can be written as

$$-x\Delta M \frac{\partial \mathbf{S}}{\partial x} = [L_1 + aL_2 + \Delta L_3 + a^2\Delta L_4]\mathbf{S}, \quad (29)$$

where M , L_1 , L_2 , L_3 , and L_4 are linear, order of unity, complex operators which depend on x and ξ . L_1 depends linearly on the frequency ω and contains $\partial/\partial\xi$ terms, while L_3 contains both $\partial/\partial\xi$ and $\partial^2/\partial\xi^2$ terms. To solve numerically this equation, the ξ dependence of $\mathbf{S}$ is discretized using a staggered Chebyshev spectral collocation technique developed by Khorrami.¹⁷ This method has the advantage of eliminating the need of two artificial pressure boundary conditions at $\xi=0$ and $\xi=\infty$, which for that reason are not included in (25)–(28). The boundary conditions at infinity (25) are applied at a truncated radial distance $\eta_{\max}=r_{\max}/\alpha(z)$, chosen large enough to ensure that the results do not depend on that truncated distance. The computations showed that $\eta_{\max}=30$ ($\xi_{\max}=900$) was sufficient for most profiles to obtain an accuracy of six significant figures. Since the value of $\eta_{\max}$ does not affect to the computation time much, $\eta_{\max}=50$ ($\xi_{\max}=2500$) was used in most of the reported computations. To implement the spectral numerical method, Eq. (29) is discretized by expanding $\mathbf{S}$ in terms of truncated Chebyshev series. A nonuniform coordinate transformation is used to map the interval $0 \leq \xi \leq \xi_{\max}$ into the Chebyshev polynomials domain $-1 \leq s \leq 1$, $\xi = c_1(1+s)/(c_2-s)$, where c_1 is a constant ($c_1=3$ in all the computations) and $c_2=1+2c_1/\xi_{\max}$. This transformation allows large values of ξ to be taken into account with relatively few basis functions.¹⁷

The ξ domain is thus discretized in N points, N being the number of Chebyshev polynomials in which $\mathbf{S}=[iF, G, H, \Pi]^T$ has been expanded. In the results presented here, N ranged between 40 and 100. The streamwise derivative $\partial\mathbf{S}/\partial x$ is approximated by the finite difference form $(\mathbf{S}_{j+1}-\mathbf{S}_j)/(\Delta x)_j$, where j is the step index in the axial direction, and $(\Delta x)_j$ the step size. A marching technique is used to solve the $4N$ discretized equations resulting from (29), starting at $x=x_0$. Since the unknown a appears with $\mathbf{S}$ on the right-hand side of (29), it constitutes, with the normalization condition, a system of nonlinear equations for $\mathbf{S}$ and a . Iteration is used to solve the nonlinear system of discretized algebraic equations at each axial station $j+1$: one starts with the results of the previous station j , and uses (29) with a_j to obtain a first approximation for $\mathbf{S}_{j+1}$; these are used in the normalization condition to yield a first approximation for a_{j+1} , which is again used to correct $\mathbf{S}_{j+1}$; the iteration procedure is continued until the modifications in the real and imaginary parts of a are both less than 10^{-8} . Usually, between two and six iterations were needed. The process is repeated at the next marching step.

As the initial condition at $x=x_0$, the eigenvalues a_0 and eigenvectors $\mathbf{S}_0$ of the right-hand side of Eq. (29) equated to zero are used:

$$0=[L_1+a_0L_2+\Delta L_3+a_0^2\Delta L_4]\mathbf{S}_0. \quad (30)$$

The solution obtained with this local eigensolution as the initial condition presents a short transient region before converging to the solution of the PSE. As we shall see, this transient region is so short that there is no need to refine the initial condition with the first term of a Taylor series around $x=x_0$.⁹ The eigensolutions of (30) are equivalent to those obtained in the *temporal* stability analysis of T (see next

section). However, since in the *spatial* stability analysis performed here one fixes the (real) frequency ω of the perturbations to obtain the complex wave number a (spatial amplification γ and axial wave number α), (30) constitutes a nonlinear eigenvalue problem, instead of the linear one of the temporal analysis. This nonlinear eigenvalue problem is solved using the linear companion matrix method described by Bridges and Morris.¹⁸ The resulting (complex) linear eigenvalue problem of dimension $8N$ is solved with the IMSL subroutine DGVCCG, which provides the entire eigenvalue and eigenvector spectrum. Owing to nonlinearity, the size of the matrices in the spatial eigenvalue problem is thus twice the size of the matrices in the temporal analysis for the same value of N , and the computation time is much larger. Also, due to the large dimensions of the matrices in (30), a relatively large amount of spurious numerical eigenvalues with very small wave numbers (large wavelengths) are produced by the eigenvalue solver, particularly when ω^* is also very small. They are easily discarded, however, because the corresponding growth rates increase without bound with N , instead of rapidly converging to a finite value, as it happens for eigenvalues of physical modes. Thus, a minimum or cutoff value of α is used when looking for the most unstable mode (highest γ) for a given frequency and flow parameters. This lower limit is easily selected by just increasing N . Finally, the axial step size $(\Delta x)_j$ is adjusted according to the local wavelength of the perturbation. A fraction e of the local wavelength, $(\Delta x)_j=e\Delta_0/\alpha_j$, is selected at each axial step to meet a given tolerance.

III. RESULTS

The results presented here are for a Long's vortex ($m=1$) with three different values of the nondimensional axial velocity at the axis: $A_1=1.2$, corresponding to a Type I solution with $M \approx 6.838$; $A_1=-0.5$, which corresponds to a Type II solution with $M \approx 5.76$, and $A_1=0.15$, which approximately corresponds to the minimum or folding value of the flow force $M=M^* \approx 3.75$ (see T for a plot of the velocity profiles). All the results are for the most unstable modes (highest γ), for a selection of frequencies (ω), azimuthal wave numbers (n), and Reynolds numbers ($\Delta^{-1}=\Delta_0^{-1}$) at different axial locations. The azimuthal wave numbers considered are $n=0$ (axisymmetric perturbations), $n=-1$, and $n=+1$. As shown in the references cited in Sec. I, these last modes with $|n|=1$ are the most relevant spiral modes. Results for $n<-1$ and $n>1$ are qualitatively equivalent to those for $n=-1$ and $n=+1$, respectively.

A. Local nonparallel results

As mentioned above, the eigenvalues and eigenfunctions of the local problem (30) are used as the initial condition for the PSE (29). Before presenting some representative local results, it is convenient to check the numerical implementation of the linear companion matrix method used to solve (30) by comparing these spatial results with those obtained from the equivalent temporal stability analysis of T. According to Gaster,⁷ given a perturbation of the form $\exp[i(\alpha x - \omega t)]$, with complex wave number $\alpha = \alpha_r + i\alpha_i$ and complex

TABLE I. Comparison between temporal (T) and spatial (S) stability results for $A_1 = 1.2$, $n = -1$, $\Delta_0 = 10^{-3}$, and several values of α . $N = 40$ in both (T) and (S), and $x = 1$ in (S).

$\alpha(T)$	$\omega_r(T)$	$\omega_i(T)$	$c_g(T)$	$\omega_i(T)/c_g(T)$	$\gamma(S)$	$\alpha(S)$
0.340	0.170	0.178	1.371	0.130	0.132	0.339
0.480	0.428	0.242	1.832	0.132	0.136	0.478
0.770	1.039	0.193	2.265	0.0832	0.0880	0.771

frequency $\omega = \omega_r + i\omega_i$, the ratio between the temporal, $\omega_i(T)$, and spatial, $-\alpha_i(S)$, growth rates is equal to the group velocity, $c_g = \partial\omega_r/\partial\alpha_r$, provided that the growth rates are small and that the frequency ω_r and the wave number α_r are both equal in the temporally and spatially increasing disturbances [(T) and (S) denote results from the temporal and spatial stability analysis, respectively]. In the notation of the present work and of T, one has that $\gamma(S) = \omega_i(T)/c_g(T)$, provided that $\alpha(S) = \alpha(T)$ and $\omega_r(T) = \omega(S)$, and $x = 1$ in (S). Table I compares the temporal and spatial stability results for a Type I vortex with $A_1 = 1.2$ and $\Delta = 10^{-3}$ when non-axisymmetric disturbances with $n = -1$ and different wave numbers α are present in the flow. One selects first a value of $\alpha(T)$ and obtains from the analysis in T $\omega_i(T)$, $\omega_r(T)$, and $c_g(T) = \partial\omega_r(T)/\partial\alpha(T)$. Then, $\omega(S)$ is set equal to $\omega_r(T)$ in the present eigenvalue problem (30) with $x = 1$ to obtain the spatial growth rate $\gamma(S)$ and axial wave number $\alpha(S)$. Taking into account the disparity of the methods used to solve the linear (T) and the nonlinear (S) eigenvalue problems, and that the growth rates are not so small, the agreement is good.

Figures 1–3 show the local growth rates and axial wave numbers for the most unstable modes at $x = 1$ as functions of the frequency for the three selected basic flows and values of

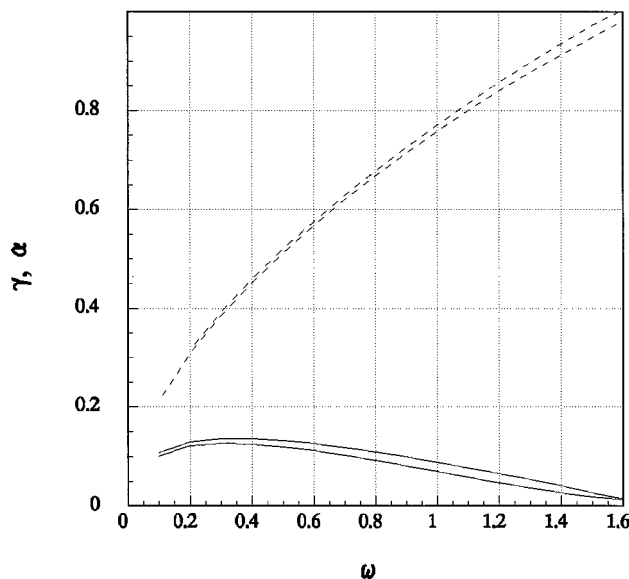


FIG. 1. Local growth rates γ (continuous lines) and axial wave numbers α (dashed lines) for the most unstable modes with azimuthal wave number $n = -1$ at $x = 1$ as functions of the frequency for a Type I flow with $A_1 = 1.2$ and two Reynolds numbers: $\Delta = 10^{-3}$ (upper curves) and $\Delta = 10^{-2}$ (lower curves). $N = 40$.

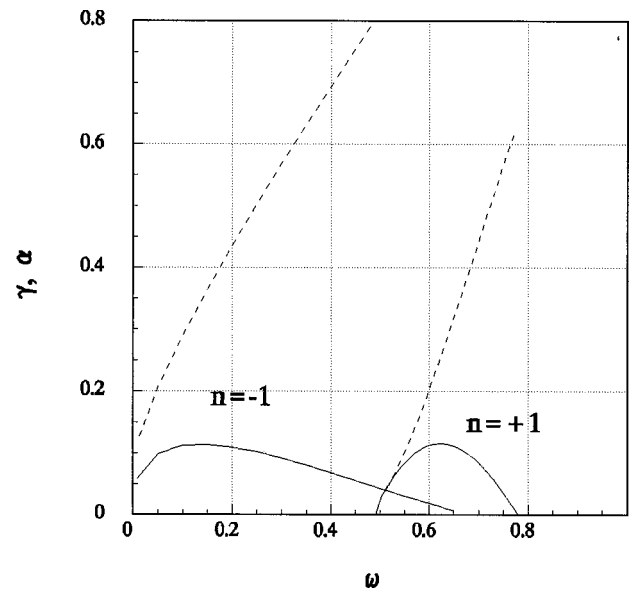


FIG. 2. Local growth rates $\gamma(\omega)$ (continuous lines) and axial wave numbers $\alpha(\omega)$ (dashed lines) for the most unstable modes with $n = -1$ and $n = 1$ at $x = 1$ for a flow with $A_1 = 0.15$ and $\Delta = 10^{-3}$. $N = 60$ for $n = -1$ and $N = 100$ for $n = 1$.

n . In agreement with T, flows corresponding to Type I solutions are unstable only for non-axisymmetric disturbances with negative azimuthal wave numbers ($n = -1$ in Fig. 1). Type II flows are also unstable for non-axisymmetric disturbances with $n > 0$, and, more importantly, for axisymmetric disturbances [Fig. 3(a) shows the results for $n = -1$, while Fig. 3(b) those for $n = 1$ and $n = 0$]. Finally, flows corresponding to the folding value $A_1 = 0.15$ are unstable for non-axisymmetric perturbations with negative and positive values of n ($n = -1$ and $n = 1$ in Fig. 2). The number N of Chebyshev polynomials used in the computations increases from Fig. 1 to 3 because, as shown in Fig. 4 for $n = -1$, the number of local maxima and minima in the most unstable eigenfunctions increases as A_1 decreases; that is, the eigenfunctions for a Type II flow are more “complex” than for a Type I flow, and a higher resolution in the radial direction is needed to obtain a comparable accuracy in the results (larger radial truncation values $\xi_{\max}$ are also needed for the basic flow). One can observe in Figs. 1–3 that the most unstable modes have usually a growth rate γ with a maximum less than 0.2 for a particular frequency, and an axial wave number α which increases almost linearly with the frequency. Thus, the phase speed $c = \omega/\alpha$ and the group velocity $c_g = \partial\omega/\partial\alpha$ of these modes are both positive and almost frequency independent. They therefore correspond to convective instabilities. However, this is not the case for the most unstable modes with $n = -1$ in a Type II flow, where γ is quite larger and α decreases as ω increases [see Fig. 3(a), where also shown is the second most unstable mode, which has a similar pattern to that of the most unstable modes for Type I flows]. The most remarkable feature of these highly unstable modes is their negative group velocity, which explains why they were not found in the temporal stability analysis of T: the corresponding temporal growth rate $\omega_i(T) = \gamma(S)c_g$ is also negative. Thus, the most unstable

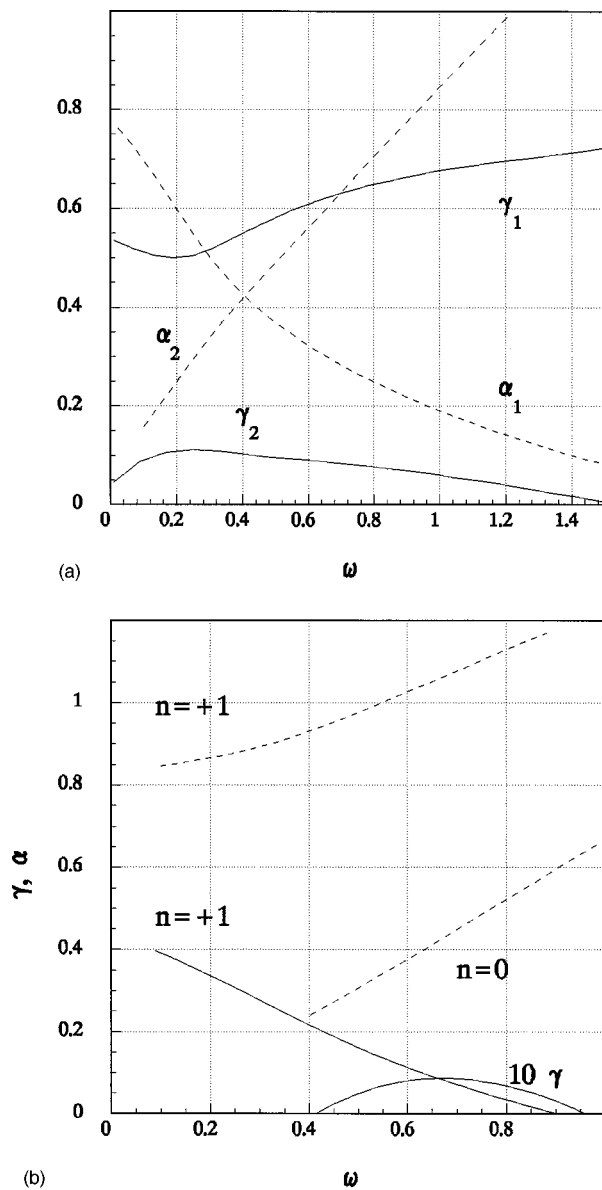


FIG. 3. Local $\gamma(\omega)$ (continuous lines) and $\alpha(\omega)$ (dashed lines) at $x=1$ for a Type II flow with $A_1 = -0.5$ and $\Delta = 10^{-3}$. (a) $n = -1$. The two most unstable modes are shown with subscripts 1 and 2, respectively. (b) $n = 1$ and $n = 0$ (note that for $n = 0$ the plotted growth rate is multiplied by 10). $N = 100$ in all cases.

temporal modes found in T for $n < 0$ in a Type II flow correspond to the spatial modes labelled with the subscript 2 in Fig. 3(a). The present computations show that the unstable spiral modes with negative group velocities appear for $A_1 < 0$, with $c_g \rightarrow 0^-$ for $A_1 \rightarrow 0^-$. Thus, most Type II Long's vortices (those with negative axial velocity at the axis) are *subcritical* in Benjamin's sense:¹⁹ they sustain both upstream- and downstream-traveling waves, that is, linearly unstable modes with both negative and positive group velocities. Type I vortices are *supercritical* because they can sustain only downstream-traveling waves, that is, convectively unstable modes with positive group velocities.

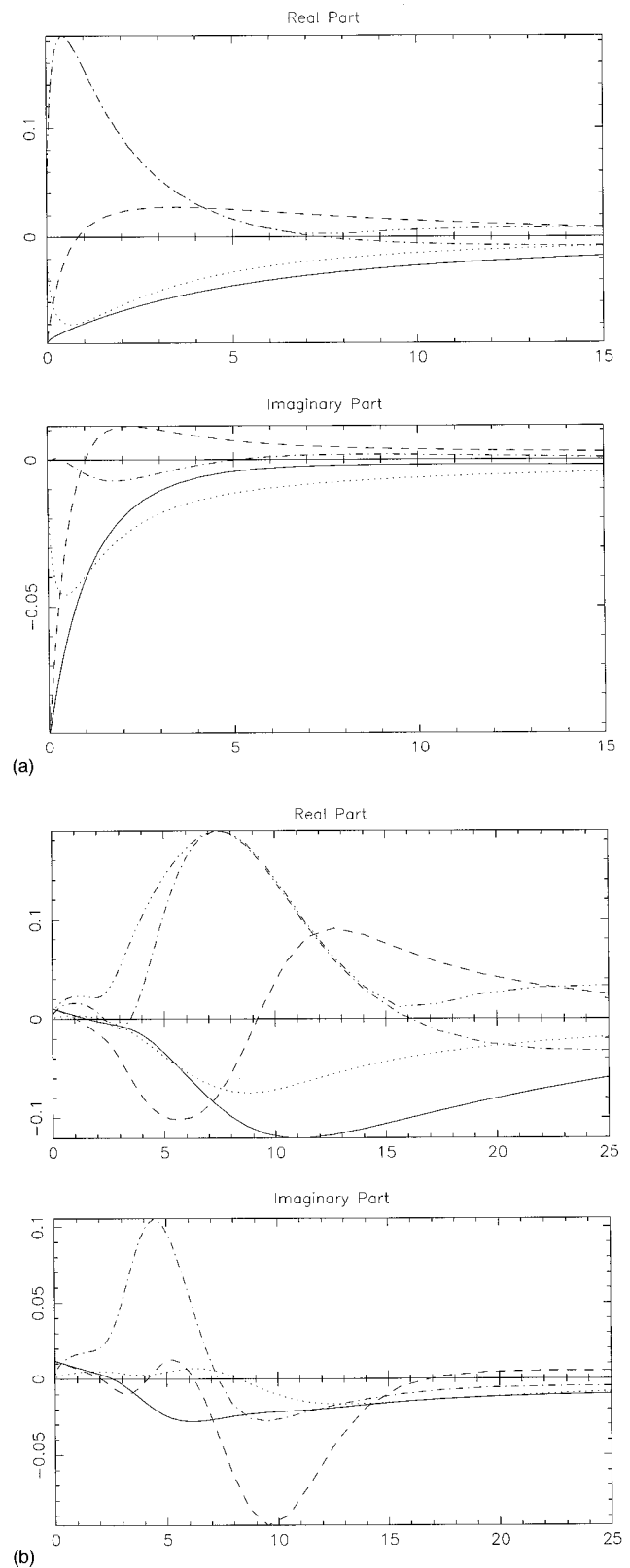


FIG. 4. Real and imaginary parts of the eigenfunctions F (full lines), G (dashed lines), H (dash-dot-dash lines), and Π (dotted lines) for the most unstable modes with $n = -1$ and $\omega = 0.3$ at $x = 1$. Also included along with the real parts is $|H|$ (dash-dot-dot-dot lines). (a) $A_1 = 1.2$, $N = 40$. (b) $A_1 = 0.15$, $N = 80$.

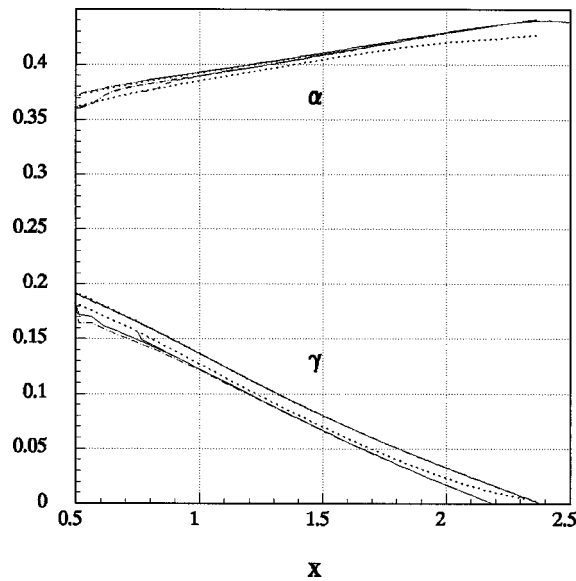


FIG. 5. Axial evolution of the growth rate γ (continuous lines) and axial wave number α (dashed lines) of the most unstable mode with $n = -1$ for $A_1 = 1.2$, $\omega = 0.3$, and two Reynolds numbers corresponding to $\Delta_0 = 10^{-3}$ (upper curves) and $\Delta_0 = 10^{-2}$. These results are obtained from the PSE using the normalization condition based on a_2 , while the dash-dot-dash lines correspond to the same cases but using the normalization condition based on a_1 . In the first case, the PSE are integrated starting at both $x_0 = 0.5$ and $x_0 = 0.75$. Finally, the dotted lines correspond to the eigenvalues of (30) at each x . $N = 50$.

As observed in Fig. 3(b), the maximum growth rate is also larger than 0.2 for non-axisymmetric disturbances with $n > 0$ in a Type II flow ($n = +1$ in that figure), reaching a maximum as $\omega \rightarrow 0$, where $\alpha(\omega)$ is no longer a linear function, but with the group velocity always positive. In the same figure it is also observed that the unstable axisymmetric disturbances have a growth rate more than an order of magnitude smaller than the non-axisymmetric ones.

B. PSE results

The numerical integration of the PSE (29) is started at some axial location x_0 using the eigenvalues and eigenfunctions of Eq. (30) as the initial condition. To test the appropriateness of this initial condition, Fig. 5 shows the axial evolution of the growth rate and axial wave number of the most unstable mode for $A_1 = 1.2$, $n = -1$, and two values of the Reynolds number, starting at two different axial locations: $x_0 = 0.5$ and $x_0 = 0.75$. For the higher Reynolds number considered ($\Delta = 10^{-3}$), the integration from $x = 0.75$ cannot be distinguished from that starting at $x_0 = 0.5$, and both results are also indistinguishable from the local ones. For the lower Reynolds number ($\Delta = 10^{-2}$), for which the differences with the local eigenvalues are larger, the results of the integration from $x_0 = 0.75$ with the local eigenvalue also coincides practically with the ones from $x_0 = 0.5$ after a short transient interval whose length is just a few axial steps of the numerical integration. This behavior is also observed for all the values of A_1 , n , and ω considered.

Figure 5 compares also the PSE results obtained with the two different normalization conditions discussed in Sec. II B.

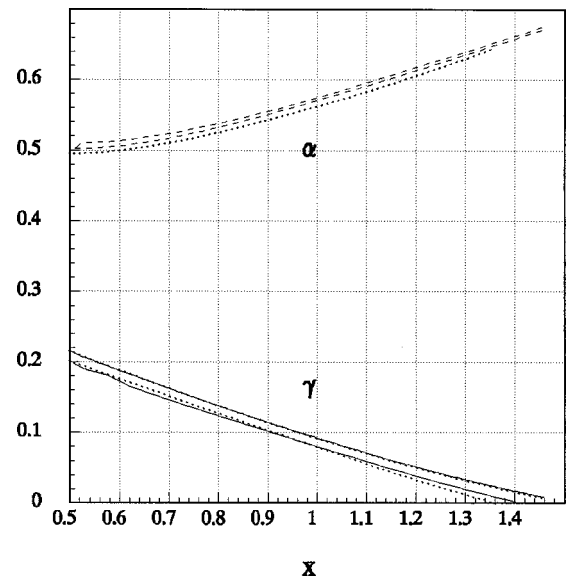


FIG. 6. $\gamma(x)$ (continuous lines) and $\alpha(x)$ (dashed lines) for the most unstable mode corresponding to $A_1 = 0.15$, $n = -1$, $\omega = 0.3$, and two values of Reynolds number: $\Delta_0 = 10^{-3}$ (upper curve for γ and lower one for α ; $N = 50$), and $\Delta_0 = 10^{-2}$ ($N = 60$). Also included are the corresponding eigenvalues of (30) (dotted lines).

Since for the Type I flow considered in that figure $|H|$ has only a maximum in the radial direction [see Fig. 4(a)], both results almost coincide. However, as A_1 decreases, the number of local maxima of $|H|$ increases [Fig. 4(b)], and the normalization condition based on $\bar{a}_1$ yields poorer results. In particular, for Type II flows, the absolute maximum of $|H|$ may shift at some axial location, producing an artificial jump in the results. For this reason, all the results given below are obtained with the normalization condition based on $\bar{a}_2$, which makes use of a physical growth rate defined in terms of the radial integral of the axial velocity of the disturbances.

One can see in Fig. 5 that the local results coincide with the results from the PSE for high Reynolds numbers (e.g., $\Delta = 10^{-3}$). The differences increase with Δ , particularly for small growth rates γ . Thus, for $\Delta = 10^{-2}$ in Fig. 5, the axial location where γ vanishes predicted by the local formulation is larger than the obtained from the PSE. However, except for these very small values of γ , the differences between local and PSE results remain small. For non-axisymmetric disturbances with $n = -1$ when $A_1 = 0.15$ and $A_1 = -0.5$ (Figs. 6 and 7), and for $n = 1$ when $A_1 = -0.5$ (Fig. 8), the differences are even smaller than in Fig. 5. In all these cases, the growth rate γ decays axially from its starting value at $x_0 = 0.5$, until it eventually vanishes at some axial location. In addition to n and the basic flow (A_1), the location where γ vanishes depends on the frequency ω , which in the reported results is chosen such that this location is the farthest (approximately). This pattern of the solution is less clear in Fig. 7, where the computations have to be stopped when the axial wave number α becomes too small. However, this exceptional situation occurs only for the unstable modes with negative group velocities corresponding to $n = -1$ in a Type II flow with a two-cell pattern ($A_1 < 0$). For these disturbances with negative group velocities, the streamwise marching of the PSE method is not physically appropriate.

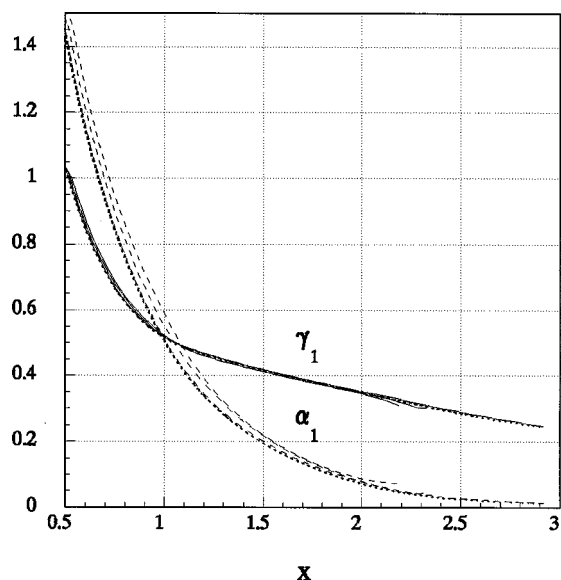


FIG. 7. $\gamma(x)$ (continuous lines) and $\alpha(x)$ (dashed lines) for the most unstable modes corresponding to $A_1 = -0.5$, $n = -1$, $\omega = 0.3$, and three values of the Reynolds number: from top to bottom $\Delta_0 = 10^{-2}$, 5×10^{-3} , and 10^{-3} . The dotted lines correspond to the eigenvalues of (30), which practically coincide for the three Reynolds numbers. $N = 60$.

For disturbances with $n = +1$ in a flow with $A_1 = 0.15$ (Fig. 9), and $n = 0$ with $A_1 = -0.5$ (Fig. 10), the situation is quite different. The flow is stable at $x_0 = 0.5$, becoming unstable at some axial location, where γ becomes positive. Then, γ reaches a maximum, and decreases until it vanishes at some further axial location. Thus, for these disturbances, the corresponding Long's vortices are convectively unstable only in a definite axial region, rather than being unstable "until" some axial location, as it happens in all the other cases considered. This situation is more in accordance with

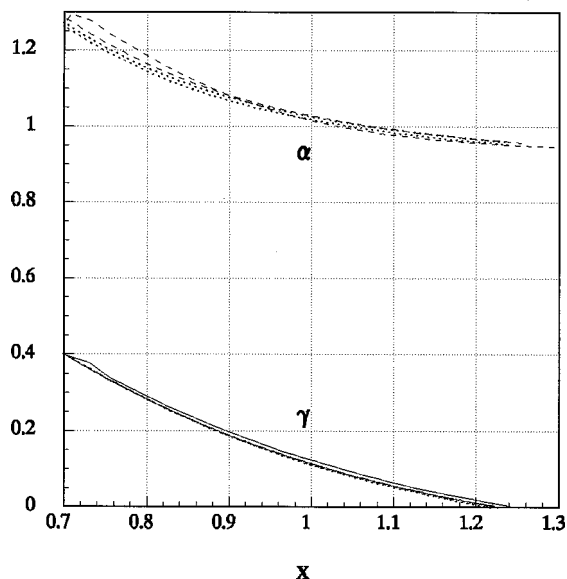


FIG. 8. $\gamma(x)$ (continuous lines) and $\alpha(x)$ (dashed lines) for the most unstable modes corresponding to $A_1 = -0.5$, $n = +1$, $\omega = 0.6$, and two values of Reynolds number: $\Delta_0 = 10^{-2}$ (upper curves), and 10^{-3} . Also included are the corresponding eigenvalues of (30) (dotted lines). $N = 60$.

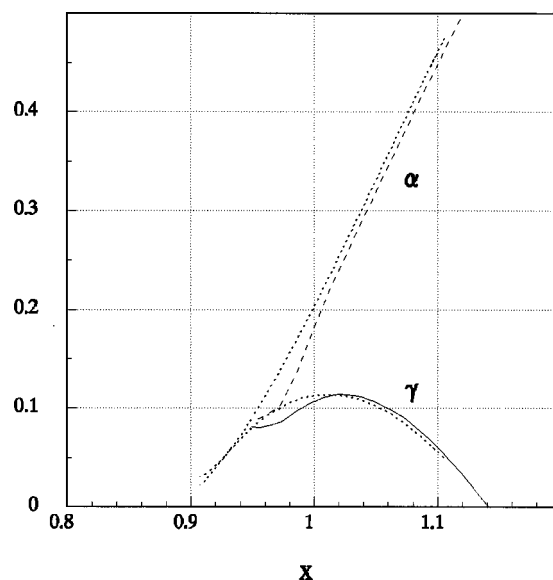


FIG. 9. $\gamma(x)$ (continuous line) and $\alpha(x)$ (dashed line) for the most unstable modes corresponding to $A_1 = 0.15$, $n = +1$, $\omega = 0.6$, and $\Delta_0 = 10^{-3}$. The dotted lines correspond to the eigenvalues of (30). $N = 60$.

the convective instabilities found, for instance, in boundary layer flows [but, in the present case, the local Reynolds number is constant along the flow, $\Delta(x) = \Delta_0$ for $m = 1$]. The differences between local and PSE results are also comparatively larger than in the other cases considered. It must be also noted that these disturbances become stabilized by viscosity at higher Reynolds numbers than those shown in Figs. 5–8, so that only the value $\Delta = 10^{-3}$ is plotted (for $\Delta = 10^{-2}$ these two disturbances are stable, see below).

Finally, in order to compare the streamwise variation of the most unstable modes for the different Long's vortices and azimuthal wave numbers considered, and their stabilization due to viscosity, it is instructive to represent the axial evolution of the amplitude of the perturbations, with refer-

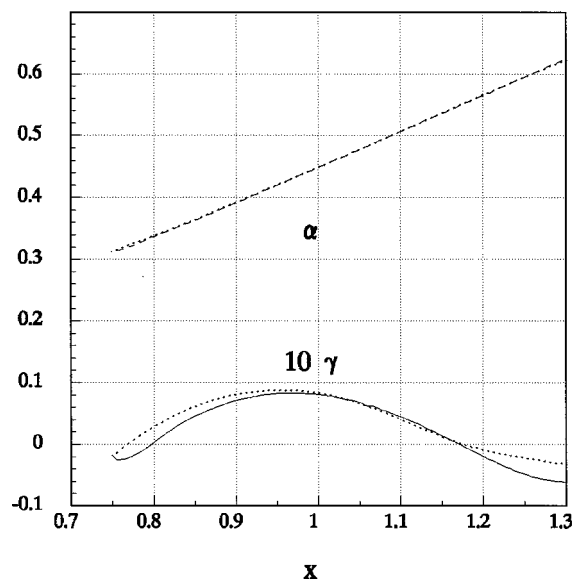


FIG. 10. $10 \times \gamma(x)$ (continuous line) and $\alpha(x)$ (dashed line) for the most unstable modes corresponding to $A_1 = -0.5$, $n = 0$, $\omega = 0.7$, and $\Delta_0 = 10^{-3}$. The dotted lines correspond to the eigenvalues of (30). $N = 60$.

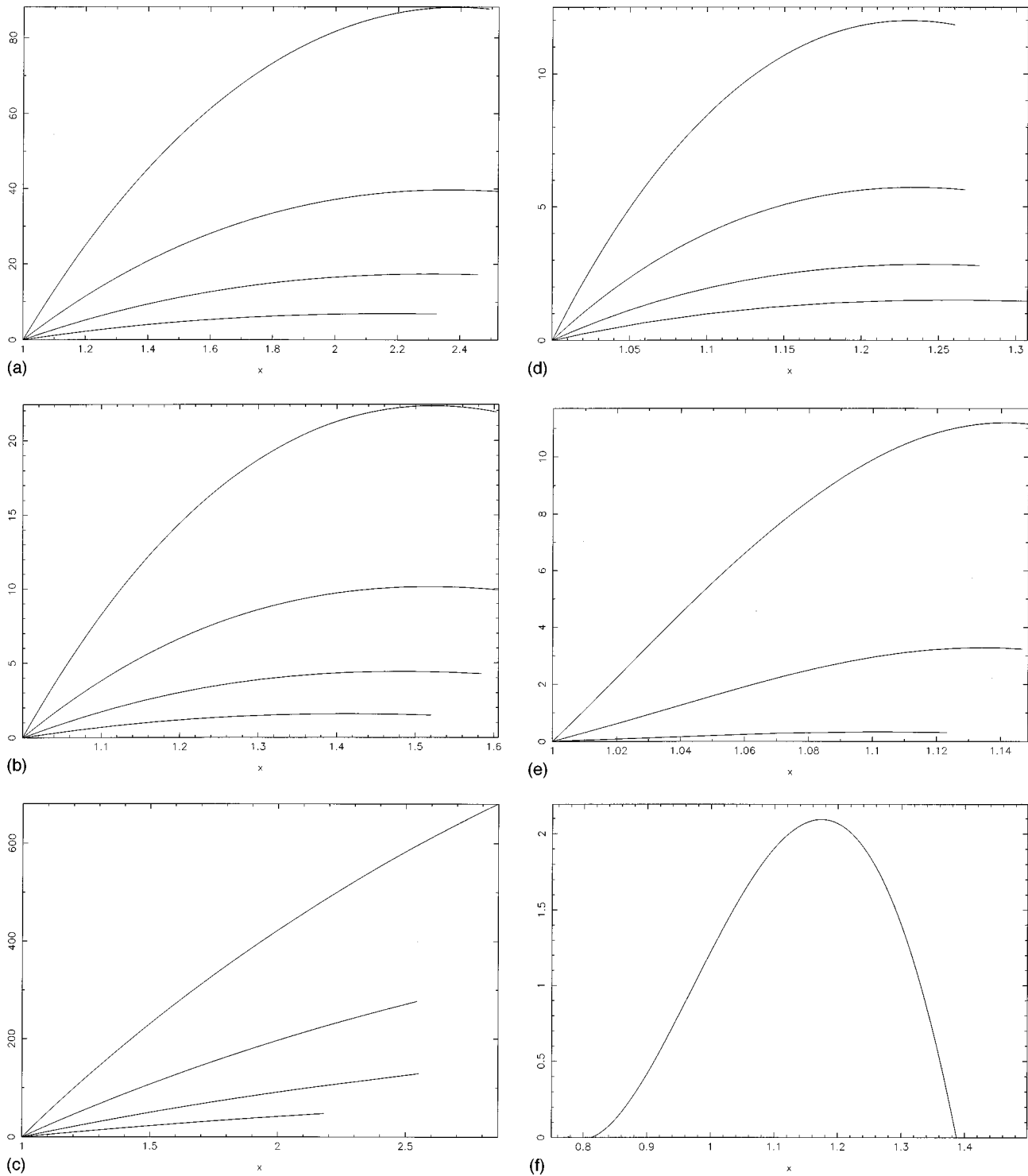


FIG. 11. $A(x)$ for, from top to bottom, $\Delta_0 = 10^{-3}$, 2.15×10^{-3} , 4.64×10^{-3} , and 10^{-2} , in the following cases: (a) $A_1 = 1.2$, $n = -1$, $\omega = 0.3$ ($N = 50$); (b) $A_1 = 0.15$, $n = -1$, $\omega = 0.3$ ($N = 60$); (c) $A_1 = -0.5$, $n = -1$, $\omega = 0.3$ ($N = 60$); (d) $A_1 = -0.5$, $n = +1$, $\omega = 0.6$ ($N = 60$); (e) $A_1 = 0.15$, $n = +1$, $\omega = 0.6$ ($N = 100$); and (f) $A_1 = -0.5$, $n = 0$, $\omega = 0.7$ ($N = 80$).

ence to the amplitude at a given location ($x=1$, say), for each case at different Reynolds numbers. To that end, Figs. 11(a)–11(f) show the logarithm of the amplification A based on the axial velocity of the perturbation,

$$A(x) \equiv \text{Real} \left[\frac{\tilde{w}(x)}{\tilde{w}(x_0=1)} \right] = \frac{1}{\Delta_0} \int_1^x \gamma(x') dx', \quad (31)$$

for the different values of A_1 , n , and ω considered, and for decreasing values of the Reynolds number. In particular, the series $\text{Re} = \Delta_0^{-1} = 10^3$, 464, 215, and 100 has been selected. As expected from (31), the amplification increases almost linearly with Re . However, as Re decreases, the disturbances eventually become stabilized, as it is clear for an axisym-

metric disturbance when $A_1 = -0.5$ [Fig. 11(f)], where only the case $Re=10^3$ is unstable, and for $n = +1$, $A_1 = 0.15$ [Fig. 11(e)], where the disturbance is stable for $Re=100$. Note that for the axisymmetric disturbance [Fig. 11(f)], the plotted amplitude starts at the beginning of the convective instability, rather than at $x_0=1$. For a given Re , the maximum amplification depends not only on the magnitude of the amplification rate γ , but also on the axial location where γ vanishes. Thus, although the values of γ at $x=1$ for $n = -1$ and $\omega = 0.3$ are of the same order of magnitude for basic flows with $A_1 = 1.2$ and $A_1 = 0.15$, the maximum amplification is about one order of magnitude larger in the former case because γ vanishes farther away [Figs. 11(a) and 11(b)]. Note also that the results plotted in Fig. 11(c) for $A_1 = -0.5$ and $n = -1$, corresponding to the highly unstable modes with negative group velocity, show very large amplifications because γ , and the location where γ vanishes, are both large.

IV. SUMMARY AND CONCLUSIONS

The spatial, nonparallel linear stability of Long's vortex has been analyzed using parabolized stability equations. These equations result naturally from the linear stability formulation of the high Reynolds number problem when one retains terms up to order $Re^{-1} = \Delta$, which is the same degree of approximation of the basic flow. The PSEs are solved with a marching technique in the axial direction combined with a staggered Chebyshev spectral collocation method in the radial direction. The results are compared with local ones from the eigenvalue problem of the local spatial stability formulation. Nonparallelism is partially included in this local formulation: it accounts for the effect of the streamwise variation of the basic flow at the same order of magnitude as the effect of viscosity on the stability of the flow. However, the local equations do not take into account the effect of the history of the disturbance on its stability, which is also $O(\Delta)$. The comparison provides thus a quantification of this effect on the stability of an important class of vortices. It is shown that the differences in the growth rate and the axial wave number are, in general, very small even for moderately low Re ($Re \sim 10^2$), below which the unstable disturbances become usually stabilized by viscosity. The differences may, however, be larger for small growth rates, which are better predicted from the PSE. Thus, the effect of the history of the disturbances is important at low Re to accurately predict the axial location where an unstable mode becomes stabilized, and therefore to accurately obtain neutral curves of stability. For the same reason, it is also important to accurately predict the onset of new convectively unstable modes, as shown for non-axisymmetric modes with positive n when A_1 is near the folding value of M , and, more importantly, for unstable axisymmetric modes for Type II vortices, which have growth rates more than an order of magnitude smaller than the helical modes. To these considerations one has to add the important advantage of the PSE of being computationally much faster to solve than the nonlinear eigenvalue problem of the local spatial stability analysis, particularly when a high radial resolution, and therefore a high value of N , is needed. How-

ever, a local eigenvalue problem has to be solved always to provide for the initial condition in the marching method used to solve the PSE.

The present spatial stability analysis confirms, for convective instabilities, the linear stability pattern of Long's vortex found with the temporal stability analysis of T: Both Type I ($A_1^* \approx 0.15 < A_1 < \infty$) and Type II ($-1/\sqrt{2} < A_1 < A_1^*$) Long's vortices are unstable to non-axisymmetric disturbances with $n < 0$, while only Type II flows are convectively unstable for axisymmetric disturbances in certain frequency ranges and axial locations. These axisymmetric unstable modes have growth rates much smaller than the non-axisymmetric ones. Helical disturbances with $n > 0$ become unstable when A_1 decreases below a certain value a little larger than A_1^* . For a given axial location x , all these inviscid instabilities become stable as Re decreases below a critical value which depends on the basic flow (A_1) and on the perturbation (n and ω), except for Type II flows with A_1 near its minimum value (ring-jet vortex), for which purely viscous unstable modes may appear below a certain value of Re (see T).

The local spatial analysis reveals, however, the existence of new helical, counter-rotating, unstable modes for Type II flows which are not found with the temporal computations owing to their negative group velocities. These new unstable modes, which have a much larger spatial growth rate than the other inviscid convectively unstable modes, show that Types II and I flows are fundamentally different from a stability point of view not only because the former ones are convectively unstable to axisymmetric ($n=0$) disturbances, but also because only Type II flows can sustain upstream-traveling waves (with $n = -1$). Therefore, Type II Long's vortices are subcritical in Benjamin's sense, or absolutely unstable for disturbances with $n = -1$, to use the more recent concepts of absolute/convective instabilities,²⁰ which have been very recently applied to the stability analysis of Batchelor's vortex.²¹ Type I vortices are just convectively unstable for non-axisymmetric disturbances (mainly with negative azimuthal wave numbers, except very close to the folding value of the flow force), and, therefore, they are supercritical swirling flows. Actually, the transition between supercritical and subcritical Long's vortices takes place at $A_1 = 0$, rather than at $A_1 = A_1^* \approx 0.15$, i.e., when the axial velocity at the axis becomes zero, in qualitative agreement with Benjamin's theory on vortex breakdown.¹⁹

The present spatial stability analysis is also more appropriate than the temporal counterpart to study the convective instabilities of Long's vortex because it provides naturally the streamwise evolution of the disturbances, this being so with independence of the fact that the effect of the history of the perturbations on their stability cannot be correctly taken into account in a temporal stability analysis. From the results one observes that all the inviscid unstable modes eventually become stable at a downstream axial location which depends on the basic flow and on the upstream perturbation considered, a consequence of the nonparallelism of the basic flow, with a vortex core radius increasing with x . At the axial location where the growth rate γ vanishes for a given perturbation, the amplitude A reaches a maximum and then decays

(Fig. 11). However, the maximum of the amplitude is usually so large that one would have to consider nonlinear effects before reaching that axial location to correctly predict the subsequent streamwise evolution of the linearly unstable perturbation.

ACKNOWLEDGMENTS

This research has been carried out with the support of the Ministerio de Educación y Cultura of Spain, Grant No. PB96-0679-C02-01, and of a NATO Collaborative Research Grant, CRG 950368. The author is grateful to Dr. I. G. Loscertales for checking the equations.

APPENDIX

Some considerations about the integrated axial momentum flux of the basic vortex are made in this appendix.

Multiplying the axial momentum equation

$$U \frac{\partial W}{\partial r} + W \frac{\partial W}{\partial z} + \frac{\partial P/\rho}{\partial z} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial W}{\partial r} \right)$$

by r , integrating across the vortex, and making use of continuity, one obtains

$$\int_0^{K\delta(z)} \frac{\partial}{\partial z} \left(W^2 + \frac{P}{\rho} \right) r dr = \nu \left[r \frac{\partial W}{\partial r} \right]_{r=K\delta(z)} - (rUW)_{r=K\delta(z)}, \quad (A1)$$

where K is a large positive constant and $\delta(z)$ is the boundary layer thickness (5). Taking into account that¹⁴ $f(\xi) \sim (C\xi)^{m/2} [1 + Q\xi^{\lambda_-}]$ for $\xi = K^2 \gg 1$, where C and Q are integration constants and λ_- is the negative root of $\lambda^2 + (m-1)\lambda + m - 2 + (m-1)L^2/2 = 0$, according to (4) and (5), the right-hand side terms in (A1) vanishes as K^{m-2} and $K^{2(m-1+\lambda_-)}$, respectively (note that for $m=1$, $\lambda_- = -1$, and the last term goes to zero as K^{-2}). Therefore, the integral on the left-hand side of (A1) vanishes for $K \rightarrow \infty$. Performing the differentiation and substituting (3) and (4), one finds

$$\begin{aligned} & \int_0^{K\delta(z)} \frac{\partial}{\partial z} \left(W^2 + \frac{P}{\rho} \right) r dr \\ &= \frac{d}{dz} \int_0^{K\delta(z)} \left(W^2 + \frac{P}{\rho} \right) r dr - K^2 \delta'(z) \delta(z) \\ & \quad \times \left(W^2 + \frac{P}{\rho} \right)_{r=K\delta(z)} \\ &= \frac{d}{dz} \left(\frac{\nu z^2}{2\delta^2} \right) \int_0^{K^2} (4f'^2 + \beta) d\xi \\ & \quad - K^2 \frac{\nu^2 z}{m\delta^2} [4f'^2 + \beta]_{\xi=K^2} \rightarrow 0 \quad \text{as } K \rightarrow \infty. \end{aligned}$$

Thus, on using (5) and the large- ξ behaviors of f and β ,¹⁴ this last expression becomes

$$\begin{aligned} & \left(\frac{W_0}{m} \right)^{2/m} \nu (\nu z)^{1-2/m} \left[\frac{m-1}{m} \int_0^{K^2} (4f'^2 + \beta) d\xi \right. \\ & \quad \left. - mC^m \left(1 - \frac{L^2}{2(2-m)} \right) K^{2(m-1)} \right] \rightarrow 0, \quad K \rightarrow \infty. \quad (A2) \end{aligned}$$

Since the first term vanishes for $m=1$ (Long's vortex), this constraint can be satisfied only if $L=\sqrt{2}$, as it was shown from a first integral of the self-similar axial momentum equation in Ref. 14, and it is also found numerically. Note that if $L=\sqrt{2}$, the pressure term cancels the W^2 term at large ξ , and the integral in (A2) becomes bounded as $K \rightarrow \infty$ for $m=1$. Thus, the nondimensional flow force, defined as $M \equiv 2\pi \int_0^\infty (W^2 + P/\rho) r dr / (W_0 m L C^{m/2})^2 = \pi \int_0^\infty (4f'^2 + \beta) / (m L C^{m/2})^2$, is finite for Long's vortex.¹³ For $1 < m < 2$, that integral is unbounded, but the constraint (A2) is satisfied because constant C is such that this infinity cancels with the also unbounded second term inside the square brackets. As it is shown in Ref. 14, this cancellation is only possible for values of L below a critical value $L^*(m)$, which is smaller than $\sqrt{2}$ for $1 < m < 2$. For $0 < m < 1$, the second term in (A2) becomes zero, and so must be the integral as $K \rightarrow \infty$, thus selecting the value of C . This is shown to be possible for L above a critical value $L^*(m) > \sqrt{2}$.

It is worth noticing that the flow force is an invariant of the motion for $0 < m \leq 1$:

$$\frac{d}{dz} \int_0^\infty \left(W^2 + \frac{P}{\rho} \right) r dr = 0 \quad \text{for } 0 < m \leq 1.$$

In fact, the flow force is a nonvanishing constant only for $m=1$ (Long's vortex): for $0 < m < 1$ it is zero, while for $1 < m < 2$ it is infinity. In general, for $0 < m < 2$, the solution satisfies the more general integral constraint

$$\lim_{K \rightarrow \infty} \int_0^{K\delta(z)} \frac{\partial}{\partial z} \left(W^2 + \frac{P}{\rho} \right) r dr \rightarrow 0. \quad (A3)$$

¹M. R. Foster and P. W. Duck, "The inviscid stability of Long's vortex," *Phys. Fluids* **25**, 1715 (1982).

²M. R. Foster and F. T. Smith, "Stability of Long's vortex at large flow force," *J. Fluid Mech.* **206**, 405 (1989).

³K. Ardlan, K. Draper, and M. R. Foster, "Instabilities of the Type I Long's vortex at large flow force," *Phys. Fluids* **7**, 365 (1995).

⁴M. R. Khorrami and P. Trivedi, "The viscous stability analysis of Long's vortex," *Phys. Fluids* **6**, 2623 (1994).

⁵M. R. Foster and D. Jacqmin, "Non-parallel effects in the stability of Long's vortex," *J. Fluid Mech.* **244**, 289 (1992).

⁶R. Fernandez-Feria, "Viscous and inviscid instabilities of non-parallel self-similar axisymmetric vortex cores," *J. Fluid Mech.* **323**, 339 (1996).

⁷M. Gaster, "A note on the relation between temporally-increasing and spatially-increasing disturbances in hydrodynamic stability," *J. Fluid Mech.* **14**, 222 (1962).

⁸A local spatial stability formulation of Long's vortex is also given by P. G. Drazin, W. H. H. Banks, and M. B. Zaturka, "The development of Long's vortex," *J. Fluid Mech.* **286**, 359 (1995). However, these authors do not report numerical stability results to compare with.

⁹F. P. Bertolotti, Th. Herbert, and P. R. Spalart, "Linear and nonlinear stability of the Blasius boundary layer," *J. Fluid Mech.* **242**, 441 (1992).

¹⁰F. P. Bertolotti, "Transition modelling based on the PSE," in *Transition and Turbulence Modelling*, edited by Henningson *et al.* (Kluwer, Dordrecht, 1996).

¹¹Th. Herbert, "Parabolized stability equations," *Annu. Rev. Fluid Mech.* **29**, 245 (1997).

¹²R. R. Long, "Vortex motion in a viscous fluid," *J. Met.* **15**, 108 (1958).

- ¹³R. R. Long, "A vortex in an infinite fluid," *J. Fluid Mech.* **11**, 611 (1961).
- ¹⁴R. Fernandez-Feria, J. Fernandez de la Mora, and A. Barrero, "Solution breakdown in a family of self-similar nearly-inviscid axisymmetric vortices," *J. Fluid Mech.* **305**, 77 (1995).
- ¹⁵O. R. Burggraf and M. R. Foster, "Continuation and breakdown in tornado-like vortices," *J. Fluid Mech.* **80**, 685 (1977).
- ¹⁶G. K. Batchelor and A. E. Gill, "Analysis of the stability of axisymmetric jets," *J. Fluid Mech.* **14**, 529 (1962).
- ¹⁷M. R. Khorrami, "A Chebyshev spectral collocation method using a staggered grid for the stability of cylindrical flows," *Int. J. Numer. Methods Fluids* **12**, 825 (1991).
- ¹⁸T. J. Bridges and P. J. Morris, "Differential eigenvalue problems in which the parameter appears nonlinearly," *J. Comput. Phys.* **55**, 437 (1984).
- ¹⁹T. B. Benjamin, "Theory of the vortex breakdown phenomenon," *J. Fluid Mech.* **14**, 529 (1962).
- ²⁰P. Huerre and P. A. Monkewitz, "Local and global instabilities in spatially developing flows," *Annu. Rev. Fluid Mech.* **22**, 473 (1990).
- ²¹I. Delbende, J.-M. Chomaz, and P. Huerre, "Absolute/convective instabilities in the Batchelor vortex: a numerical study of the linear impulse response," *J. Fluid Mech.* **355**, 229 (1998).