Stability of the boundary layer flow on a long thin rotating cylinder

M. A. Herrada,¹ C. Del Pino,² and R. Fernandez-Feria²

¹Escuela Superior de Ingenieros, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

²E. T. S. Ingenieros Industriales, Universidad de Málaga, Pza. El Ejido s/n, 29013 Málaga, Spain

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The development and stability of the boundary layer flow over a long thin cylinder aligned with the main flow and which rotates around its axis is considered. Numerical results show that the introduction of rotation has an important effect on the behavior of the basic flow. When the swirl increases, the shear stress at the wall also increases due to the changes in the pressure distribution along the cylinder surface. A nonparallel linear stability analysis of the basic flow is performed using parabolized stability equations. Even at moderately low rotation, we find the existence of unstable centrifugal modes, in addition to the shear ones found in previous stability analysis of the boundary layer flow on a cylinder with no rotation. These centrifugal instabilities develop at Reynolds numbers, based on the cylinder radius and external axial velocity, much smaller than those required for the growing of the shear instabilities. Our analysis shows that nonparallel effects play a key role in the onset and development of these instabilities, being the spiral mode with azimuthal wavenumber n=1, the first to become unstable as the Reynolds number is increased in most cases of interest. We characterize the critical Reynolds number for convective instability as a function of the axial distance to the leading edge for several values of the swirl parameter. © 2008 American Institute of Physics. [DOI: 10.1063/1.2885330]

I. INTRODUCTION

The boundary layer flow over a long thin cylinder with zero pressure gradient has been the subject of several studies.¹⁻⁵ This axisymmetric flow undergoes viscous instabilities similar to the Tollmien–Schlichting waves in two-dimensional boundary layers over a flat plate, which were fully characterized recently by Tutty *et al.*⁶ Earlier stability analyses were made by Morris and Byon,⁷ and by Kao and Chow.⁸ As an important difference with Blasius flow, however, curvature effect makes several nonaxisymmetric modes unstable at lower critical Reynolds numbers than axisymmetric ones, but becomes stable again sufficiently far downstream.

Kao and Chow⁸ also considered the case of the boundary layer flow on a rotating circular cylinder and found that the flow becomes unstable at lower Reynolds numbers than in the nonrotating case, shifting the onset of instability toward the leading edge of the cylinder. These authors carried out a temporal stability analysis employing a Chebyshev collocation spectral method, in one of the first instances that this method was used for linear stability analyses, considering only two low values of the swirl parameter.

In addition to its fundamental interest, the flow over a long cylinder (with or without rotation) may be viewed as an idealization of the general flow along a thin body of revolution (spinning or not), for which it is of practical interest to characterize the onset of instability. Several types of these instabilities were experimentally visualized and discussed by Mueller *et al.*⁹ As we shall see below, for slender cylinders, nonparallel effects are very important to characterize the onset and development of these instabilities, especially for rotating cylinders and for nonaxisymmetric perturbations, the

more so the larger the spinning rate and the thinner the cylinder. For these reasons we undertake here a spatial, nonparallel stability analysis of the boundary layer flow over a rotating cylinder, characterizing the critical Reynolds numbers and locations for the onset of the instabilities (both axisymmetric and nonaxisymmetric) in a wide range of values of the swirl parameter. Thus, the present work complements those of Tutty et al.⁶ and Kao and Chow⁸ in a double way: it supplies the frequencies for the unstable modes by using a spatial analysis, which are more useful from a practical point of view than the wavenumbers obtained by the temporal analyses, and, more important, corrects the instability characteristics by taking into account nonparallel effects. We pay special attention to the rotating cylinder because nonparallel effects are more important in that case, and because the work by Kao and Chow for the rotating cylinder is quite less complete than that by Tutty et al. for the nonrotating case. In particular, Kao and Chow considered only very low values of the rotation rate, and did not characterize the low wavenumber (low frequency) limit, so that in these cases even our parallel results are also new. In addition, the results of Kao and Chow for the nonrotating cylinder are in clear disagreement with those by Tutty et al.,⁶ probably due to the limitations in the wavenumber of the perturbations. For these reasons no comparison with the results by Kao and Chow is given here. We shall see that our parallel stability results for a nonrotating cylinder practically coincide with the results by Tutty et al.,⁶ but these results are substantially modified when nonparallel effects are taken into account. The nonparallel stability code used in this work has been previously tested with experimental results for the developing flow in a rotating cylinder.¹⁰

The general problem formulation and the numerical method are described in Sec. II. Numerical results on the flow stability are given and discussed in Sec. III. Finally, a summary of the results and the conclusions are given in Sec. IV.

II. FORMULATION OF THE PROBLEM AND NUMERICAL TECHNIQUES

We consider the incompressible flow of a fluid of density ρ and kinematic viscosity ν over a semi-infinite circular cylinder of radius *R*. We use cylindrical coordinates, (r, θ, z) , with the axial coordinate *z* along the axis of the cylinder and the origin at its leading edge. The uniform external flow, aligned along the cylinder axis, reaches the leading edge with a uniform axial velocity W_{∞} . In addition, the cylinder may rotate around its axis at an angular velocity Ω .

The velocity field $\mathbf{u} \equiv (u, v, w)$ is made dimensionless with the freestream velocity W_{∞} , the radial coordinate *r* is nondimensionalized with the radius of the cylinder *R*, while a characteristic length *L* is used to render dimensionless the axial coordinate *z*. Finally, time *t* is made dimensionless with the characteristic time R/W_{∞} , and pressure *p* with ρW_{∞}^2 . Consequently, the problem is governed by three dimensionless parameters: a Reynolds number based on the freestream velocity and the radius of the cylinder,

$$\operatorname{Re} = \frac{W_{\infty}R}{\nu},\tag{1}$$

an aspect ratio

$$\Delta = \frac{R}{L},\tag{2}$$

and a swirl parameter

$$S = \frac{\Omega R}{W_{\infty}}.$$
(3)

We assume that the radius *R* of the cylinder is small relative to the characteristic axial length *L*, $\Delta \ll 1$. Also, we assume that the Reynolds number is large, in such a way that Re $R/L \sim O(1)$, i.e., that *L* is the axial length over which viscous effects develop along the cylinder. For simplicity, we shall select *L* such that Re $\Delta = 1$, so that Δ (or Re) disappears from the problem.

To study the evolution and stability of the flow, we represent the velocity and pressure fields as the sum of a steady axisymmetric mean flow and a small unsteady perturbation,

$$\begin{pmatrix} u(r,\theta,z,t)\\ v(r,\theta,z,t)\\ w(r,\theta,z,t)\\ p(r,\theta,z,t) \end{pmatrix} = \begin{pmatrix} \Delta U(r,z)\\ V(r,z)\\ W(r,z)\\ P(r,z) \end{pmatrix} + \begin{pmatrix} u'(r,\theta,z,t)\\ v'(r,\theta,z,t)\\ w'(r,\theta,z,t)\\ p'(r,\theta,z,t) \end{pmatrix},$$
(4)

where capital letters are used for the basic flow, and primed variables for the perturbations (of course, all the variables are dimensionless). Note that the radial component of the basic flow has been rescaled with Δ , so that U in Eq. (4) is also

order unity, according to the continuity equation. Substituting Eq. (4) into the incompressible Navier–Stokes equations we obtain separate equations for the basic flow and for the perturbations.

A. Leading order mean flow equations

Neglecting terms $O(\Delta^2)$, the mean basic flow is governed, at the leading order, by the boundary layer equations

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{\partial W}{\partial z} = 0,$$
(5)

$$\frac{V^2}{r} = \frac{\partial P}{\partial r},\tag{6}$$

$$\frac{U}{r}\frac{\partial}{\partial r}(rV) + W\frac{\partial V}{\partial z} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\frac{V}{r}\right),\tag{7}$$

$$U\frac{\partial W}{\partial r} + W\frac{\partial W}{\partial z} = -\frac{\partial P}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right).$$
(8)

These equations are solved subjected to the following boundary conditions: at the cylinder surface, r=1,

$$U(1,z) = W(1,z) = 0, \quad V(1,z) = S;$$
(9)

far away from the cylinder, $r \rightarrow \infty$,

$$U(\infty, z) = V(\infty, z) = 0, \quad W(\infty, z) = 1;$$
(10)

finally, at the cylinder edge, z=0, a uniform axial velocity and zero azimuthal velocity are assumed,

$$W(r,0) = 1, \quad V(r,0) = 0.$$
 (11)

To solve Eqs. (5)–(8) with boundary conditions (9)–(11) we have used a standard explicit method of lines. For that purpose, we have first eliminated the pressure in Eqs. (5)–(8) by adding the result of taking derivatives with respect to *z* in Eq. (6) to that obtained by taking derivatives with respect to *r* in Eq. (8). The resulting system of equations is then discretized in the radial direction using second order central differences in a nonuniform mesh with *N* points which are obtained by mapping the interval $1 \le r \le r_{\text{max}}$, where $r_{\text{max}} \gg 1$ is the radial distance where we imposed conditions (10), into the domain $-1 \le \xi \le 1$ using the algebraic transformation

$$r_i = 1 + \frac{c_1(1+\xi_i)}{c_2 - \xi_i}, \quad i = 1, \dots, N,$$
 (12)

where $\xi_i = -1 + (i-1)/(N-1)$, i=1, ..., N. Here c_1 is a free constant and $c_2 = 1 + 2c_1/(r_{max} - 1)$. We have selected $c_1 = 5$ and $r_{max} = 100$ in all the cases computed in this work. With this discretization, and after some algebra, one can obtain a simple tridiagonal system to solve the values of the radial velocity at the different lines, $(U^i)_{i=1}^N$, for each *z*, in terms of $(V^i)_{i=1}^N$ and $(W^i)_{i=1}^N$. In this manner, one arrives at a system of 2*N* ordinary differential equations for $(\partial V/\partial z)_{i=1}^N$ and $(\partial W/\partial z)_{i=1}^N$, which can be solved, for example, with a standard variable step-size fourth order Runge–Kutta method. In this problem, we have considered N=250 lines, and a rela-

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tive error tolerance of 10^{-8} for the Runge–Kutta solver, which allow for a very stable and accurate downstream integration of the boundary layer equations.

B. Parabolized stability equations

The perturbations $\mathbf{s} = [u', v', w', p']^T$ are decomposed in the standard form

$$\mathbf{s}(r,\theta,z,t) = \mathbf{S}(r,z)\chi(z,\theta,t),\tag{13}$$

where the complex amplitude

$$\mathbf{S}(r,z) \equiv \begin{pmatrix} iF(r,z) \\ G(r,z) \\ H(r,z) \\ \Pi(r,z) \end{pmatrix}$$
(14)

is allowed to depend on the axial coordinate z, in addition to the radial one, to account for the nonparallelism of the basic flow. The other part of the perturbation is an exponential that describes the wavelike nature of the disturbances,

$$\chi(z,\theta,t) = \exp\left[\frac{1}{\Delta}\int_{z_0}^z a(z')dz' + in\theta - i\omega t\right],$$
(15)

where z_0 is the axial point in which the disturbances are introduced (e.g., $z_0=0$), a(z) is the nondimensional (complex) axial wavenumber, *n* is the azimuthal wavenumber, and ω is the nondimensional frequency of the disturbances. *a* and ω are defined as

$$a \equiv iR\tilde{k} \equiv \gamma + i\alpha, \tag{16}$$

$$\omega \equiv \frac{\tilde{\omega}R}{W_{\infty}},\tag{17}$$

where \tilde{k} and $\tilde{\omega}$ are the dimensional axial wavenumber and frequency, respectively. The real part of a(z), $\gamma(z)$, is the exponential growth rate, and its imaginary part, $\alpha(z)$, is the axial wavenumber. In the spatial stability analysis to be considered here, one fixes a real frequency ω and looks for complex values of a(z). The flow is (convectively) unstable when $\gamma(z) > 0$. Finally, the azimuthal wavenumber *n* is equal to zero for axisymmetric perturbations, and different from zero for nonaxisymmetric perturbations.

Substituting Eqs. (13) and (14) into the incompressible Navier–Stokes equations, neglecting second-order terms in the small perturbations (linear stability) and terms $O(\Delta^2)$, i.e., neglecting terms with second order axial derivatives, which constitutes the basis of the parabolized stability equations (or PSE) technique,^{11,12} one obtains the following parabolic stability equation for **S**:

$$\mathbf{L} \cdot \mathbf{S} + \Delta \mathbf{M} \cdot \frac{\partial \mathbf{S}}{\partial z} = \mathbf{0}, \tag{18}$$

$$\mathbf{L} \equiv \mathbf{L}_1 + a\mathbf{L}_2 + \Delta\mathbf{L}_3 + a^2\Delta\mathbf{L}_4,\tag{19}$$

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$$\mathbf{L}_{1} = \begin{pmatrix} \frac{1}{r} + \frac{\partial}{\partial r} & \frac{in}{r} & 0 & 0\\ i\left(\frac{nV}{r} - \omega\right) & -\frac{2V}{r} & 0 & \frac{\partial}{\partial r}\\ \frac{\partial V}{\partial r} + \frac{V}{r} & i\left(\frac{nV}{r} - \omega\right) & 0 & \frac{in}{r}\\ \frac{\partial W}{\partial r} & 0 & i\left(\frac{nV}{r} - \omega\right) & 0 \end{pmatrix}, \quad (20)$$

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$$\mathbf{L}_{2} = \mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ W & 0 & 0 & 0 \\ 0 & W & 0 & 0 \\ 0 & 0 & W & 1 \end{pmatrix},$$

$$\mathbf{L}_{4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
(21)

$$\mathbf{L}_3 = \mathbf{L}_{31} + \mathbf{L}_{32},\tag{22}$$

$$\mathbf{L}_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -D_r^2 + \frac{n^2 + 1}{r^2} & \frac{2in}{r^2} & 0 & 0 \\ -\frac{2in}{r^2} & -D_r^2 + \frac{n^2 + 1}{r^2} & 0 & 0 \\ 0 & 0 & -D_r^2 + \frac{n^2}{r^2} & 0 \end{pmatrix},$$

$$D_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r},$$
(23)

$$\mathbf{L}_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial U}{\partial r} + U \frac{\partial}{\partial r} & 0 & 0 & 0 \\ 0 & U \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) & \frac{\partial V}{\partial z} & 0 \\ 0 & 0 & \left(\frac{\partial W}{\partial z} + U \frac{\partial}{\partial r} \right) & 0 \end{pmatrix}.$$
 (24)

The term corresponding to \mathbf{L}_3 has been decomposed into two terms to account for the two different physical phenomena involved: viscous effects (\mathbf{L}_{31}), and nonparallelism of the basic flow (\mathbf{L}_{32}), both of order Δ (remember that $\Delta = \mathbf{R}e^{-1}$).

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This equation has to be solved with homogeneous boundary conditions for the perturbation velocity (F=G=H=0) at both r=1 and $r \rightarrow \infty$. It also needs an *initial* condition at $z=z_0$. A convenient choice is the solution of the local eigenvalue problem^{10,11,13}

$$\mathbf{L}_0 \cdot \mathbf{S}_0 \equiv [\mathbf{L}_1 + a_0 \mathbf{L}_2 + \Delta \mathbf{L}_3 + a_0^2 \Delta \mathbf{L}_4] \cdot \mathbf{S}_0 = \mathbf{0}, \qquad (25)$$

that provides the initial eigenvalue $a_0 \equiv a(z_0)$, and eigenfunction $\mathbf{S}_0(r) \equiv \mathbf{S}(r, z_0)$, which will be used to start the axial integration of Eq. (18) for a given set of nondimensional parameters. Equation (25) accounts for the effect of the nonparallelism of the basic flow, but neglects the effect of the history or convective evolution of the perturbations. Its solution for different values of $z > z_0$ will be compared with the solution to the PSE (18) to measure the importance of this last effect. This local solution will be termed near-parallel (NP) solution. It will also be compared to the local parallel (P) solution, which coincides with (25) except for the last term proportional to Δ . In fact, these local and parallel results will be used as the reference ones to start the PSE analysis (see Sec. III).

C. Normalization condition and numerical method

As it stands, there is some ambiguity in the partition of the perturbations (13) into two functions of z. To close the problem one has to enforce an additional condition which puts some restriction on the axial variation of **S**. Basically, one uses a normalization condition that restricts rapid changes in z of **S**, according to the slow axial variation of the basic flow (small Δ). Thus, the growth rate and the axial sinusoidal variation are represented by the exponential function χ . Several types of normalization conditions can be used.^{10–13} Here we will use an integral condition based on the kinetic energy of the perturbations. Defining a physical amplification rate a_1 based on the axial variation of the kinetic energy of the perturbations,

$$a_{1}(z) \equiv \gamma_{1}(z) + i\alpha_{1}(z)$$

$$\equiv \Delta \frac{\int_{1}^{\infty} \left[u'^{\dagger} \frac{\partial u'}{\partial z} + v'^{\dagger} \frac{\partial v'}{\partial z} + w'^{\dagger} \frac{\partial w'}{\partial z} \right] dr}{\int_{1}^{\infty} \left[|u'|^{2} + |v'|^{2} + |w'|^{2} \right] dr} = a(z)$$

$$+ \Delta \frac{\int_{1}^{\infty} \left[F^{\dagger} \frac{\partial F}{\partial z} + G^{\dagger} \frac{\partial G}{\partial z} + H^{\dagger} \frac{\partial H}{\partial z} \right] dr}{\int_{1}^{\infty} \left[|F|^{2} + |G|^{2} + |H|^{2} \right] dr}, \qquad (26)$$

where \dagger denotes the complex conjugate, the normalization condition used here can be expressed as $a_1(z)=a(z)$ for all $z > z_0$. That is, at each axial step in the integration of Eq. (18), the second term in the right-hand side of Eq. (26) (the one multiplied by Δ) is set equal to zero, transferring the main part of the streamwise variation of the perturbations to the exponential function χ .

To numerically solve Eq. (18) together with its normalization condition, the radial dependence of S is discretized

using a staggered Chebyshev spectral collocation technique developed by Khorrami,¹⁴ where the three velocity components and the three momentum equations are discretized at the grid collocation points whereas the pressure and the continuity equation are enforced at the midgrid points. This method has the advantage of eliminating the need for two artificial pressure boundary conditions at r=1 and $r \rightarrow \infty$. To implement the spectral numerical method, Eq. (18) is discretized by expanding S in terms of a truncated Chebyshev series. The boundary conditions at infinity are applied at a truncated radial distance r_{max} , chosen large enough to ensure that the results do not depend on that truncated distance. To map the interval $1 \le r \le r_{max}$ into the Chebyshev polynomials domain $-1 \le s \le 1$, we use the same transformation Eq. (12), but substitute ξ_i by the Gauss–Lobatto points $s_i = \cos(\pi i/N)$, $i=0,\ldots,N$ ¹⁵ Now c_1 is a constant such that approximately half of the nodes are concentrated in the interval $1 \le r \le 1$ $+c_1$. This transformation allows large values of r to be taken into account with relatively few basis functions. We have used c_1 between 2 and 5, $r_{max} = 100$, and N between 70 and 120 in the computations reported below.

The streamwise variation of Eq. (18) is solved numerically using an implicit finite difference scheme,

$$\mathbf{L}_{j+1} \cdot \mathbf{S}_{j+1} + \Delta \mathbf{M} \cdot \frac{\mathbf{S}_{j+1} - \mathbf{S}_j}{(\delta z)_j} = \mathbf{0},$$
(27)

where j is the step index in the axial direction, and $(\delta z)_i$ the step size. A marching technique is used to solve the 4N discretized equations resulting from Eq. (27), starting at $z=z_0$. Since the unknown a appears with S, this equation constitutes, together with the normalization condition, a system of nonlinear equations for S and a. Iterations are used to solve the nonlinear system of discretized algebraic equations at each axial station j+1: one starts with the results of the previous station j, and uses Eq. (27) with a_i to obtain a first approximation for S_{j+1} ; these are used in the normalization condition to yield a first approximation for a_{i+1} , which is again used to correct S_{i+1} ; the iteration procedure is continued until the modifications in the real and imaginary parts of a are both less than a given tolerance (10^{-8}) . Usually, between 2 and 4 iterations were needed (except in the first step after $z=z_0$, where more iterations are sometimes required). The process is repeated at the next marching step. Numerical instability puts a lower limit to the axial step size $(\delta z)_i$ for given values of the physical parameters and N. This limitation strongly affects the axial accuracy of the function a(z) obtained numerically. To have some control on the numerical instability we have used the technique described by Anderson *et al.*,¹⁶ which allows the use of smaller step sizes δz in numerically stable schemes and, consequently, improving the axial accuracy of the solution. We have used values of δz between 0.005 and 0.07. It is worth mentioning here that a former version of the present PSE code was checked against experimental results by Imao et al.¹⁷ for the developing flow in a rotating pipe.¹⁰

Finally, the nonlinear eigenvalue problem (25), whose solution is used as the initial condition of Eq. (18) and its normalization condition at $z=z_0$, is solved using the linear



FIG. 1. Skin friction $\tau_T(x)$ (a), and boundary layer thickness $\delta_T(x)$ (b) for Re=10⁴ and S=0.

companion matrix method described by Bridges and Morris,¹⁸ after discretizing the radial derivatives by the Chebyshev spectral collocation method described above. The resulting (complex) linear eigenvalue problem is solved with double precision using an eigenvalue solver from the IMSL library (subroutine DGVCCG), which provides the entire eigenvalue and eigenvector spectrum. *Spurious* eigenvalues are discarded by comparing the computed spectra for increasing number N of collocation points.

III. RESULTS

The results presented here are mainly intended to characterize the influence of rotation on the structure and stability of the boundary layer flow over a long thin cylinder.

A. Boundary layer solution

At the leading order given by Eqs. (5)–(11), the boundary layer structure depends only on the swirl parameter S. To characterize the results we use the dimensionless skin friction τ , which in our formulation is given by

$$\tau(z) = \left. \frac{\partial W}{\partial r} \right|_{r=1}.$$
(28)

Another quantity of interest is the nondimensional boundary layer thickness, $\delta(z)$, defined here as the distance form the cylinder surface to the position where condition

$$W[r = \delta(z) - 1, z] = 0.99 \tag{29}$$

is met.

First we compare our results with the nonspinning case (S=0) considered by Tutty *et al.*⁶ These authors use an axial coordinate *x* related to our *z* by x=z Re; their skin friction and boundary layer thickness (denoted here with a subscript *T*) are related to Eqs. (28) and (29) through

$$\tau_T(x = \operatorname{Re} z) = \frac{\tau(z)}{\sqrt{\operatorname{Re}}}, \quad \delta_T(x = \operatorname{Re} z) = \delta(z)\sqrt{\operatorname{Re}}.$$
 (30)

Note that the Reynolds number, which does not appear in our notation for the basic flow, act just as a scaling factor for the axial coordinate and the flow properties. Figure 1 shows these quantities as functions of *x* computed for $Re=10^4$. They are in excellent agreement with the numerical results by Tutty *et al.*⁶ (their Figs. 1 and 4), in spite of the quite different numerical methods used to compute the basic flow.



FIG. 2. Skin friction $\tau(z)$ (a) and boundary layer thickness $\delta(z)$ (b) for different values of *S*: *S*=0 (solid lines), *S*=1 (dashed lines), and *S*=2 (dotted lines).

As noted by Tutty *et al.*,⁶ the skin friction and the boundary layer thickness are both significantly different from those computed from Blasius' solution for a flat plate.

Our numerical results show that the skin friction in the basic flow increases with the swirl parameter *S*. This can be seen in Fig. 2(a), where we have plotted $\tau(z)$ for several values of *S*. In addition, the introduction of swirl produces a reduction of the boundary layer thickness δ [Fig. 2(b)]. This



FIG. 3. Re*(z) (a), $\omega^*(z)$ (b), and $\alpha^*(z)$ (c), for S=0 and n=1 obtained with the different approximations, P, NP, and PSE, as indicated. Also shown with stars are the results by Tutty *et al.* (Ref. 6) for Re_c(z_c), $\omega_c(z_c)$, and $\alpha_c(z_c)$.

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TABLE I. Critical Reynolds numbers Re_c at $z=z_c$ for S=0, n=0, and n=1 obtained from P, NP, and PSE approximations compared to the results by Tutty *et al.* (Ref. 6) (denoted by T). Also given are the corresponding critical frequencies ω_c and axial wavenumbers α_c .

<i>n</i> =1	Re_{c}	Z_c	ω_c	$lpha_c$
Т	1060	0.512	0.069	0.125
Р	1054	0.51	0.069	0.1253
NP	880	0.65	0.0627	0.1087
PSE	1275	0.435	0.072	0.135
<i>n</i> =0	Re_{c}	Z _c	ω_c	α_c
Т	12439	0.00378	0.865	2.73
Р	12628	0.00374	0.842	2.682
NP	12548	0.00375	0.848	2.687

behavior is due to the fact that the swirl induces a negative pressure gradient along the cylinder surface. As a consequence of this suction effect, the axial velocity increases near the wall and, therefore, the skin friction increases and the width of the boundary layer decreases. To show this, we can take the axial derivative of the radial momentum equation (6) and then integrate it with respect to r. Taking into account that far away from the cylinder the axial pressure gradient is zero, we have

$$\left. \frac{dP}{dz} \right|_{r=1} = -\int_{1}^{\infty} \frac{1}{r} \frac{dV^2}{dz} dr < 0, \tag{31}$$

since the axial derivative of V is positive due to the viscous diffusion of the azimuthal momentum generated by the rotating cylinder [see Eq. (7)]. Note also from Eq. (31) that the larger the rotation rate of the cylinder, the larger the pressure decay along the cylinder surface, since the viscous diffusion of the azimuthal momentum is proportional to the rotation rate.

B. Stability results for S=0

Before undertaking the study of the effect of the swirl on the stability of the boundary layer flow, we have considered the stability of the nonspinning case, S=0. This case has been thoroughly investigated, but using a parallel-flow approximation and a temporal stability analysis, by Tutty *et al.*⁶ We use these previous stability results to check our numerical techniques and to characterize the nonparallel effects on the stability of the flow.

More precisely, we consider in this work three different sets of (spatial) stability results. First, those coming from the parallel-flow approximation, which are obtained from Eq. (25), but neglecting nonparallel effects of the basic flow; i.e., neglecting the " Δ -term" proportional to the matrix \mathbf{L}_{32} (24). These results will be denoted by "P," and they take into account viscous effects, which are also proportional to Δ in Eq. (25) (terms associated with \mathbf{L}_{31} and \mathbf{L}_4). These parallel results are equivalent to those obtained by Tutty *et al.*⁶ for the nonspinning case but from a temporal stability analysis. Second, near-parallel stability results coming from Eq. (25), denoted by "NP," and, finally, nonparallel stability results



FIG. 4. As in Fig. 3 but for n=0. No PSE results are shown.

coming from the full parabolized stability equation (18), denoted by "PSE." It must be noted that the NP results do not constitute a rigorous approximation intermediate between the P and the PSE results, but they are retained here independently of P and PSE results because they are appropriate initial conditions in Eq. (18) to obtain the PSE results.^{10,13}

In the case S=0 the only physical mechanism for instability is the shear of the flow, similarly to the Blasius boundary layer. For sufficiently large Reynolds numbers, the flow becomes convectively unstable ($\gamma > 0$ and group velocity $c_g = \partial \omega / \partial \alpha > 0$) within a certain region relatively close to the leading edge of the cylinder. We define the critical Reynolds number at each z-station, Re^{*}(z), as the minimum Reynolds number for which the shear mode becomes neutrally stable ($\gamma=0$) at a certain frequency $\omega^*(z)$ (the critical frequency) and axial wavenumber $\alpha^*(z)$ (the critical wavenumber). Fig-



FIG. 5. Axial evolution of the growth rate (a) and axial wavenumber (b) of the shear mode for S=0, Re=950, $\omega = 0.066$, and n=+1 computed with the PSE (solid lines) and with the local NP solution (dashed lines). The different solid lines correspond to different values of z_0 (0.02, 0.03, and 0.1) and δz (0.005 and 0.01).

ure 3 shows the variations of the critical Reynolds number, critical frequency, and critical wavenumber with z for the nonaxisymmetric (spiral) mode with n=1, which is the mode that first becomes unstable as the Reynolds number is increased for each z. We show the results obtained from the P, NP, and PSE approximations. The critical Reynolds number $\text{Re}^*(z)$ has a minimum value, which we denote by Re_c , at an axial location z_c . The values of Re_c , z_c , together with the corresponding ω_c and α_c , obtained with the three different approximations, are given in Table I. We also include the values computed by Tutty *et al.*⁶ (see Table I in that reference, but using the present notation with z=x/Re).

The first thing that one may observe is that the temporal stability results by Tutty et al. (denoted by T) practically coincide with our parallel (P) spatial stability results, as indicated by the numerical values given in Table I for n=1, and by the star symbols plotted in Fig. 3, corresponding to Re_c , ω_c , and α_c . The second significant feature in Fig. 3 and Table I is that nonparallel effects for this most unstable mode with n=1 are very important indeed. For instance, Re_c computed with the P-approximation, which practically coincide with the results by Tutty et al., is 1054; those that compute with the NP-approximation is significantly lower (880), and those that are obtained with the full PSE approximation turns out to be significantly higher (1275). That is to say, nonparallel effects that also take into account the history or evolution of the perturbations as they move downstream notably raise the critical Reynolds number. They also reduce the axial extent of the instability region, as seen in Fig. 3(a). These are significantly new results that would be interesting to check experimentally. However, no such experimental data are available, to our knowledge, in the literature.

The stability results for nonaxisymmetric perturbations with n=2,3,... are qualitatively similar, but with higher (increasing with *n*) critical Reynolds numbers $\text{Re}^*(z)$ than for the mode n=1. For this reason they are not shown here. For axisymmetric perturbations (n=0), the critical Reynolds numbers are much higher (more than one order of magnitude) than for n=1. They are computed and shown in Fig. 4 and Table I because the axisymmetric instabilities are qualitatively different from the nonaxisymmetric ones. First because the critical frequencies and wavenumbers are much higher than for the nonaxisymmetric modes. Second, and more important, because this mode becomes unstable just in a narrow *pocket* located very close to the leading edge of the cylinder (note that the axial scale in Fig. 4 is much smaller than in Fig. 3). For this last reason, the critical Reynolds numbers cannot be accurately obtained from the PSE, so that Fig. 4 and Table I only show results obtained from the P and the NP approximations, which are practically coincident in this case n=0.

To understand why the PSE results are not reliable for n=0, we show in Fig. 5 how the PSE results are obtained for the case n=1 in Fig. 3. In particular, we consider the case with Re=950 and ω =0.066, and show the axial evolution of the growth rate and axial wavenumber of this spiral shear mode computed from the numerical integration of Eq. (18). We use different δz , and start the integration at different axial locations z_0 with the eigenvalues and eigenfunctions of the NP approximation (25) as the initial condition. In the figure we also show $\gamma(z)$ and $\alpha(z)$ obtained with the NP approximation (dashed lines). Although there is an initial transient of the PSE results that depends on the starting axial location z_0 and on δz , eventually all the curves collapse to yield, approximately, the same critical values, corresponding to $\gamma=0$. In all the reported PSE computations we have varied z_0 and δz to ensure that the results do not depend on these numerical parameters. In the inset of Fig. 5(a) we also observe the important differences in the critical values given in Fig. 3. For the particular case considered, the flow is unstable in the interval $0.5 \leq z \leq 0.65$, according to the NP approximation, while it is stable for all the values of z according to the PSE. Thus, nonparallel effects associated with the development of the perturbation along the cylinder play a crucial role and are even more important than nonparallel effects associated with the basic flow itself. On the other hand, it is seen that the



FIG. 6. Re^{*}(z) (a), $\omega^*(z)$ (b), and $\alpha^*(z)$ (c), for S=0.1 and n=1 obtained with the different approximations, P, NP, and PSE, as indicated.

PSE transient behavior is very important near the leading edge of the cylinder, where the axial gradients of the boundary layer flow are the greatest. Therefore, when the flow becomes unstable at very small values of z, it occurs for the axisymmetric shear mode, and the PSE cannot capture accurately the critical values of the parameters. However, we have seen that the results from both the P and the NP approximations are very close to each other when n=0 (Fig. 4), in contrast to the results for the mode n=1 (Fig. 3), so that it is expected that the PSE will not differ much from these results either.

C. Stability results for S>0

The stability of the flow is substantially altered when the cylinder rotates, even for very small rotation rates, owing to the appearance of new centrifugal modes that become con-



FIG. 7. Radial distribution of the eigenfunctions $(|H|, |F|, |G|, |\Pi|, \text{normal$ $ized to } |H|_{\text{max}}=1)$ of the most unstable modes with n=1 obtained with the P approximation at z=0.5, for S=0, Re=1200, $\omega=0.075$ (continuous lines), and for S=0.1, Re=200, $\omega=0.04$ (dashed lines).

vectively unstable at much lower Reynolds numbers than the shear modes. This is a well known fact in some other related stability problems such as the flow inside a rotating pipe, $^{10,19-22}$ or the swirling boundary layer flow inside a stationary pipe,²³ when compared to the stability of their nonrotating or nonswirling counterparts. In the present problem, Fig. 6(a) shows that for a small value of the swirl parameter such as S=0.1, centrifugal instabilities with n=1 becomes unstable at Reynolds numbers of the order of 20, if z is large enough, while the minimum value of Re^* for S=0, where only the shear modes were present, was near 1300 [from PSE in Fig. 3(a)]. Another qualitative difference between the shear modes for S=0 and the centrifugal ones for S>0 is that the minimum value of $\operatorname{Re}^*(z)$, Re_c , reached at a finite, relatively small, axial location z_c for S=0 [$z_c \approx 0.435$ from PSE in Fig. 3(a)], is now $z_c \rightarrow \infty$ for S > 0; i.e., Re* tends asymptotically to a minimum value as $z \rightarrow \infty$ for S > 0 [see Fig. 6(a)]. To better appreciate the differences between shear and centrifugally unstable modes, Fig. 7 compares the eigenfunctions of the most unstable modes with n=1 for S=0 and S=0.1 near the critical values of Re and ω corresponding to z=0.5, when computed with the parallel approximation. Apart from the greater radial extent of the velocity perturbations in the case with swirl, which is just due to the much lower Reynolds number, the main difference resides in the pressure perturbation, which for the centrifugal mode is much more concentrated near the axis and, what is more relevant for the centrifugal instability, decays almost linearly near the axis, so that the pressure gradient of the perturbation cannot support its associated centrifugal force near the axis.

It must be noted that, unlike the case S=0, n=1 is not always the first mode to become unstable as Re increases for any z. Figure 8(a) shows that, for S=0.1, n=1 is actually the mode that first become unstable as Re increases for most values of z. However, for very small z, particularly, for z $<z_{12}(S=0.1) \approx 0.09$, the azimuthal mode n=2 is the first to become unstable as Re increases. This value is so close to z=0 to be hardly relevant. Note also in Fig. 8(a) that the



FIG. 8. (a) $\operatorname{Re}^*(z)$ computed from the NP approximation for S=0.1 and for n=1,2, and 3, as indicated. (b) and (c) Axial coordinate z_{12} and Reynolds number Re_{12} at which the curves $\operatorname{Re}^*(z)$ for n=1 and n=2 cross each other as functions of *S*.

curves $\text{Re}^*(z)$ corresponding to n=1 and n=2 cross each other at $\text{Re}_{12}(S=0.1) \approx 113$, which is a relatively high value of Re compared to the minimum value of Re*, which tends to approximately 20 for large z. Therefore, instabilities with n=2 are not physically relevant in this case, except if they are expressly excited by some artificial mean. This behavior is qualitatively similar for all the values of the swirl parameter S considered here [see Figs. 8(b) and 8(c), where $z_{12}(S)$ and $\text{Re}_{12}(S)$ are plotted for $0 < S \le 1$]. In fact, $z_{12}(S)$ has a maximum at $S \approx 0.1$ [Fig. 8(b)], so that for other values of S the mode n=2 is the first to become unstable only at smaller values of z than for S=0.1. The curves $\text{Re}^*(z)$ for n>2 never cross the curves $\text{Re}^*(z)$ for n=1, so that they always become unstable at larger Reynolds numbers. On the other hand, the axisymmetric mode (n=0), and asymmetric modes with n < 0, remain of the shear type for S > 0, becoming unstable at much higher Reynolds numbers, comparable to the critical Reynolds numbers for S=0 given in the previous section. For all these reasons, only the instability results for n=1 are reported here.

In relation to the three different approximations, P, NP, and PSE, Fig. 6(a) shows that the stability predictions from PSE are between those from P and those from NP: for a given z, Re* from PSE is between the Re* computed from the parallel and near-parallel approximations. The difference in the computed values of Re*(z) decreases as z increases, but are significant for z around unity, and become very important as z goes to zero. On the other hand, for a given Re, the axial location where the flow becomes unstable, $z^*(Re)$, are quite different when computed from the three different approximations, the more so the higher z: $z^*(Re)$ from the PSE is much smaller than z^* from the parallel approximation, but it is much larger than that computed from the NP approximation for the same Re.

Figures 6(b) and 6(c) show that $\omega^*(z)$ and $\alpha^*(z)$ from PSE are both larger than the P and the NP counterparts. A curious feature of these figures is that the most unstable mode computed from the parallel approximation changes at $z \approx 1.5$, producing a jump in ω^* and α^* , but without appreciable effect in Re^{*}. This mode switching is not observed when nonparallel effects are considered in the stability formulation, neither in the NP nor in the PSE approximations.

As the swirl parameter S increases, the stability picture is qualitatively similar to that just described for S=0.1, with the mode n=1 the first to become (centrifugally) unstable as Re increases for a given z [except for very small values of z; see Fig. 8(b)]. Globally, $\operatorname{Re}^*(z)$ decreases as S increases. For S=1, Fig. 9(a) shows that Re* is around 10 for z=1, decreasing below 10 as z increases. The asymptote of Re^{*} as $z \rightarrow \infty$, which is practically the same from the three approximations, is now between 3 and 4 (it was around 20 for S=0.1). For these relatively low values of the Reynolds number, the present boundary layer and PSE approximations become poor, and one has to be cautious about these stability results for large z. But for z of order unity or smaller, Re^* is sufficiently large, even for large S, to be confident in the PSE approximation (remember also that z is scaled with Re^{-1} , so that z=1 corresponds to a distance Re times the cylinder radius from the leading edge).

The results in Fig. 9 for S=1 shows that Re^{*} does not differ much when computed from the three different approximations. However, in terms of z^* for a given Re, the results are quite different: now z^* is much larger from the PSE than from the other two approximations. In relation to the critical frequency and wavenumber, $\omega^*(z)$ and $\alpha^*(z)$, both are significantly larger when obtained from the PSE than from the P and NP approximations, whose results are very close to each other. Again, nonparallel effects of the basic flow and the axial evolution of the perturbations are crucial to detect with precision the axial location where the flow becomes unstable for a given Reynolds number, and to determine the frequency and the wavenumber of these unstable modes.



FIG. 9. As in Fig. 6 but for S=1.

IV. SUMMARY AND CONCLUSIONS

We have considered in this work the structure and stability of the boundary layer flow over a rotating cylinder. We have characterized the onset of convective instabilities taking into account nonparallel effects due to the axial evolution of both the basic flow and the perturbations by means of the PSE technique.

First we have considered the nonrotating cylinder case to compare with existing parallel-flow stability results (Tutty *et al.*).⁶ We have checked that our spatial stability analysis reproduces the temporal stability results of these authors (critical Reynolds numbers, critical frequencies, and wave-numbers) in the parallel flow approximation. However, non-parallel effects modify substantially these results, especially for the nonaxisymmetric mode with n=1, which is the first to become unstable as the Reynolds number is increased. The



FIG. 10. $\text{Re}^*(z=0.5)$ for n=1 as a function of S obtained from the three different approximations.

PSE technique shows that, for this most unstable spiral mode, the critical Reynolds number is increased, and the instability is shifted toward the leading edge, in relation to the parallel flow results. But the critical frequency and the critical wavenumber do not differ much from the parallel flow approximation.

Rotation of the cylinder increases friction and reduces the boundary layer thickness due to the change in the pressure distribution along the cylinder surface. The rotating boundary layer flow becomes now unstable to centrifugal perturbations with n > 0 at much lower Reynolds number than the shear modes present in the nonrotating case. Thus, for n=1, which again is the first mode to become unstable as the Reynolds number is increased for most relevant values of the axial coordinate, the flow becomes centrifugally unstable for Reynolds number of the order of 10, sufficiently far downstream the leading edge, even for relatively low values of the swirl parameter. Figure 10 summarizes the critical Reynolds numbers for the mode n=1 at a given axial location (z=0.5) for increasing values of S, showing the dramatic decrease of Re^{*} when the cylinder rotates (S > 0; note the logarithmic vertical scale). Nonparallel effects do not affect too much to the critical Reynolds number for a given S > 0 at a fixed axial location z, but for a given supercritical Reynolds number, the axial location where the flow becomes unstable, together with the frequency and the wavenumber of the amplified mode, are strongly affected by the nonparallelism of the basic flow and the perturbations (see Figs. 6 and 9). It would be of interest to perform experiments to test these nonparallel stability results, both for the spinning and for the nonspinning cases.

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