

The onset of absolute instability of rotating Hagen–Poiseuille flow: A spatial stability analysis

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A spatial, viscous stability analysis of Poiseuille pipe flow with superimposed solid body rotation is considered. For each value of the swirl parameter (inverse Rossby number) $L > 0$, there exists a critical Reynolds number $Re_c(L)$ above which the flow first becomes convectively unstable to nonaxisymmetric disturbances with azimuthal wave number $n = -1$. This neutral stability curve confirms previous temporal stability analyses. From this spatial stability analysis, we propose here a relatively simple procedure to look for the onset of absolute instability that satisfies the so-called Briggs–Bers criterion. We find that, for perturbations with $n = -1$, the flow first becomes absolutely unstable above another critical Reynolds number $Re_t(L) > Re_c(L)$, provided that $L > 0.38$, with $Re_t \rightarrow Re_c$ as $L \rightarrow \infty$. Other values of the azimuthal wave number n are also considered. For $Re > Re_t(L)$, the disturbances grow both upstream and downstream of the source, and the spatial stability analysis becomes inappropriate. However, for $Re < Re_t$, the spatial analysis provides a useful description on how convectively unstable perturbations become absolutely unstable in this kind of flow. © 2002 American Institute of Physics. [DOI: 10.1063/1.1497374]

I. INTRODUCTION

It was first shown by Pedley^{1,2} that Poiseuille flow, when subjected to rapid rotation about the axis of the pipe, becomes unstable to nonaxisymmetric disturbances. The temporal stability analysis, both inviscid¹ and viscous,² was later extended to cover all values of the parameters: Reynolds number Re , swirl parameter L (inverse Rossby number), azimuthal wave number n , and axial wave number α .^{3–6} In the present paper, the problem of spatial stability where a disturbance, imposed at a specific location in the fluid, grows or decay with axial distance, is investigated for axisymmetric and nonaxisymmetric disturbances. As in the temporal stability analysis, it is shown that the flow is stable to axisymmetric disturbances, and unstable to nonaxisymmetric perturbations in a wide region of the Re, L plane, $n = -1$ being the azimuthal wave number that first becomes unstable as Re , or L , is increased. In fact, as pointed out by Cotton and Salwen,⁵ the envelope of the neutral stability curves coincide for the spatial and for the temporal problems. But the spatial analysis is physically more relevant from an experimental point of view, providing the forcing frequencies at which the flow is most unstable (or least stable) for each n, Re and L . In particular, for each value of $L > 0$ [as it is widely known, the nonrotating Poiseuille flow ($L = 0$) is stable for any infinitesimal disturbance],^{7,8} the analysis provides the frequency ω_c at which the flow first becomes unstable as the Reynolds number increases above a critical value $Re_c(L)$. It is shown that for increasing $Re > Re_c(L)$, the instability is first convective, i.e., wave packets corresponding to an unstable frequency with positive spatial growth rate travel with positive real group velocity (although the real phase velocity could be positive or negative, depending on ω, Re and L), thus moving away from the source. As the Reynolds number is further

increased, the real part of the group velocity corresponding to the most unstable frequency decreases until it vanishes at a second critical Reynolds number $Re_t(L) > Re_c(L)$. This situation, which for $n = -1$ occurs only for $L > L^* \approx 0.38$, marks, as we shall show, the onset of the absolute instability of the flow according to the Briggs–Bers criterion.^{9,10} For $Re > Re_t$, unstable modes with both positive and negative group velocities coexist, and the spatial stability analysis of the present work becomes meaningless. It is concluded that the spatial stability analysis provides a very simple and useful tool to describe, in this kind of swirling flow, the transition process from convective to absolute instability as the governing parameters are varied.

There are several recent works on the convective–absolute transition in swirling jets and wakes (i.e., Batchelor and related vortex models).^{11–16} These flows are quite different to (more complex than) the one considered in this paper. They have in common that absolute instability is usually associated with wake-like axial velocity profiles. In the present work we find the transition to absolute instability in a Poiseuille axial velocity profile for relatively low values of the swirl parameter that characterizes the superimposed solid body rotation. Unfortunately, no experimental data are available (to our knowledge) to compare with. Pedley² already pointed out the difficulties in obtaining a Hagen–Poiseuille flow plus solid body rotation experimentally. Mackrodt⁴ reported some (rather qualitative) experimental results that agreed well with the theoretical convective instability results. But these experiments have not been pursued further to find out the convective–absolute transition in a fully developed rotating Hagen–Poiseuille flow. The comparison with experimental results from other types of swirling flows in pipes is not appropriate due to the strong qualitative differences between the flows.

II. VISCOUS STABILITY FORMULATION AND NUMERICAL METHOD

The basic flow considered in this work is the rotating Hagen–Poiseuille flow in a pipe of radius R , which in cylindrical polar coordinates (r, θ, z) has a velocity field given by

$$[U, V, W] = [0, L W_0 y, W_0(1 - y^2)], \quad (1)$$

where

$$y \equiv \frac{r}{R}, \quad (2)$$

is the nondimensional radial distance, W_0 is the maximum axial velocity at the axis and L is the swirl parameter,

$$L \equiv \frac{\Omega R}{W_0}, \quad (3)$$

with Ω the angular velocity of the rigid-body rotation (note that L is the inverse of the Rossby number used in some previous works on the temporal stability of this flow). The other dimensionless parameter governing the flow is the Reynolds number

$$\text{Re} \equiv \frac{W_0 R}{\nu}, \quad (4)$$

where ν is the kinematic viscosity. As in previous works on the temporal stability of this flow, sometimes it will be convenient to use a Reynolds number for the azimuthal flow, $\text{Re}_\theta \equiv \Omega R^2 / \nu = \text{Re} L$, instead of L .

To analyze the linear stability of the above base flow, the flow variables (u, v, w) and p are decomposed, as usual, into the mean part, (U, V, W) and P , and small perturbations:

$$u = U + W_0 \bar{u}, \quad (5)$$

$$v = V + W_0 \bar{v}, \quad (6)$$

$$w = W + W_0 \bar{w}, \quad (7)$$

$$\frac{p}{\rho} = \frac{1}{2} r^2 \Omega^2 + W_0^2 \bar{p}. \quad (8)$$

Since the basic flow is parallel along the axis, the dimensionless perturbations

$$\mathbf{s} \equiv [\bar{u}, \bar{v}, \bar{w}, \bar{p}]^T \quad (9)$$

are decomposed in the form

$$\mathbf{s} = \mathbf{S}(y) \exp[a \zeta + i n \theta - i \varpi \tau]. \quad (10)$$

In this expression,

$$\zeta \equiv \frac{z}{R} \quad \text{and} \quad \tau \equiv \frac{W_0 t}{R} \quad (11)$$

are the dimensionless axial coordinate and time, respectively, n is the azimuthal wave number,

$$\varpi \equiv \frac{\hat{\omega} R}{W_0} \quad (12)$$

is the (in general complex) dimensionless frequency of the perturbations, and $a/i \equiv k$, with

$$a \equiv i \hat{k} R \equiv \gamma + i \alpha, \quad (13)$$

is the complex nondimensional axial wave number. $\gamma = \Re(a)$ is the spatial growth rate, and its imaginary part α is the nondimensional axial wave number ($\hat{\omega}$ and $\hat{k}$ are the dimensional frequency and axial wave number, respectively). Finally, the complex amplitude $\mathbf{S}$ is written as

$$\mathbf{S}(y) \equiv \begin{pmatrix} F(y) \\ G(y) \\ H(y) \\ \Pi(y) \end{pmatrix}. \quad (14)$$

Substituting (10)–(14) into the incompressible Navier–Stokes equations, and neglecting second-order terms in the small perturbations, one obtains the following linear stability equation for $\mathbf{S}$:

$$\mathbf{L} \cdot \mathbf{S} = 0, \quad (15)$$

where the matrix operator $\mathbf{L}$ is defined as

$$\mathbf{L} \equiv \mathbf{L}_1 + a \mathbf{L}_2 + \frac{1}{\text{Re}} \mathbf{L}_3 + a^2 \frac{1}{\text{Re}} \mathbf{L}_4, \quad (16)$$

$$\mathbf{L}_1 = \begin{pmatrix} 1 + y \frac{d}{dy} & in & 0 & 0 \\ i(nL - \varpi)y & -2Ly & 0 & y \frac{d}{dy} \\ 2Ly & i(nL - \varpi)y & 0 & in \\ -2y^2 & 0 & i(nL - \varpi)y & 0 \end{pmatrix}, \quad (17)$$

$$\mathbf{L}_2 = \begin{pmatrix} 0 & 0 & y & 0 \\ y(1 - y^2) & 0 & 0 & 0 \\ 0 & y(1 - y^2) & 0 & 0 \\ 0 & 0 & y(1 - y^2) & y \end{pmatrix}, \quad \mathbf{L}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \\ 0 & 0 & -y & 0 \end{pmatrix}, \quad (18)$$

$$\mathbf{L}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -D_y + \frac{n^2+1}{y} & \frac{2in}{y} & 0 & 0 \\ -\frac{2in}{y} & -D_y + \frac{n^2+1}{y} & 0 & 0 \\ 0 & 0 & -D_y + \frac{n^2}{y} & 0 \end{pmatrix}, \quad (19)$$

with

$$D_y \equiv \frac{d^2}{dy^2} + \frac{d}{dy}. \quad (20)$$

This equation has to be solved with the following boundary conditions at the axis $y=0$,¹⁷ and the pipe wall $y=1$:

$y=0$

$$F=G=0, \quad dH/dy=0 \quad (n=0), \quad (21)$$

$$F \pm iG=0, \quad dF/dy=0, \quad H=0 \quad (n=\pm 1), \quad (22)$$

$$F=G=H=0 \quad (|n|>1); \quad (23)$$

$$F(1)=G(1)=H(1)=0. \quad (24)$$

In the spatial stability analysis that will be carried out here, for a given *real* frequency $\varpi=\omega$, and given the parameters Re , L , and n , the system (15)–(24) constitutes a nonlinear eigenvalue problem for the complex eigenvalue a . The flow is considered unstable when the disturbance grows with z ; i.e., when the real part of the eigenvalue, γ , is positive.

To solve (15)–(24) numerically, $\mathbf{S}$ is discretized using a staggered Chebyshev spectral collocation technique developed by Khorrami,¹⁸ where the three velocity components and the three momentum equations are discretized at the grid collocation points whereas the pressure and the continuity equation are enforced at the middle grid points. This method has the advantage of eliminating the need for two artificial pressure boundary conditions at $y=0$ and $y=1$, which are not included in (21)–(24). To implement the spectral numerical method, Eq. (15) is discretized by expanding $\mathbf{S}$ in terms of a truncated Chebyshev series. To map the interval $0 \leq y \leq 1$ into the Chebyshev polynomials domain $-1 \leq s \leq 1$, the transformation $y=(s+1)/2$ is used. This simple transformation concentrates the Chebyshev collocation points at both the axis and the pipe wall. The domain is thus discretized in N points, N being the number of Chebyshev polynomials in which $\mathbf{S}$ has been expanded. For most of the computations reported below, values of N between 40 and 50 were enough to obtain the eigenvalues with at least 12 significant digits, as it was checked for every result given below by using larger values of N . With this discretization, Eqs. (15)–(23) becomes an algebraic nonlinear eigenvalue problem which is solved using the linear companion matrix method described by Bridges and Morris.¹⁹ The resulting (complex) linear eigenvalue problem is solved with double precision using an eigenvalue solver from the IMSL library (subroutine DGVCCG), which provides the entire eigenvalue and eigen-

vector spectrum. Spurious eigenvalues were discarded by comparing the computed spectra for increasing number N of collocation points.

III. SPATIAL STABILITY RESULTS AND COMPARISON WITH PREVIOUS WORKS

For real values of $\varpi=\omega$, the governing stability equations have the symmetry property $a(\omega; n; \text{Re}, L) \mapsto a^*(-\omega; -n; \text{Re}, L)$, where the asterisk indicates the complex conjugate. Thus, if one allows for both positive and negative values of the forcing frequency ω , only the cases with nonpositive (or non-negative) azimuthal wave number n have to be considered. In this work, it will be assumed that $n \leq 0$. A spatial mode with $\omega < 0$, $n < 0$, and the eigenvalue $a = \gamma + i\alpha$ (for given values of Re and L) physically corresponds to a spatial mode with the positive frequency $-\omega$, the positive azimuthal wave number $-n$, the axial wave number $-\alpha$, and the same spatial growth rate γ .

To check the accuracy of the numerical results, a comparison is first made with the spatial stability results of Garg and Rouleau⁸ for a nonrotating pipe Poiseuille flow ($L=0$), which were obtained by these authors using a quite different numerical method. We have corroborated that the nonrotating flow is linearly stable for all values of ω , n , and Re . Using $N=40$, the computed eigenvalues coincide with the corrected values reported by Garg and Rouleau⁸ (their Tables 1 and 2 with the smallest step size) in all the 11 significant figures given by these authors for $n=0$, and in nine significant figures for $n=1$. The same excellent agreement is found with the numerical results reported by Khorrami *et al.*²⁰ for the same problem with $L=0$.

For $L>0$ and given values of the parameters we have looked for the least stable, or the most unstable, spatial modes propagating towards increasing z . That is, for each $L>0$, $\text{Re}>0$, $n \leq 0$, and a given positive, negative or zero value of ω , we have searched for the largest value of γ corresponding to a mode with a positive real part of the group velocity, which in its dimensionless form is given by

$$c_g \equiv \frac{\partial \omega}{\partial \alpha}. \quad (25)$$

If $\gamma < 0$ ($\gamma > 0$), the amplitude of the wave packet corresponding to the selected forcing frequency ω , which moves downstream at the real group velocity $c_g > 0$, will decrease (increase) with z , and the flow will be spatially stable (unstable). Thus, a spatial growth rate $\gamma > 0$ with $c_g > 0$ cor-

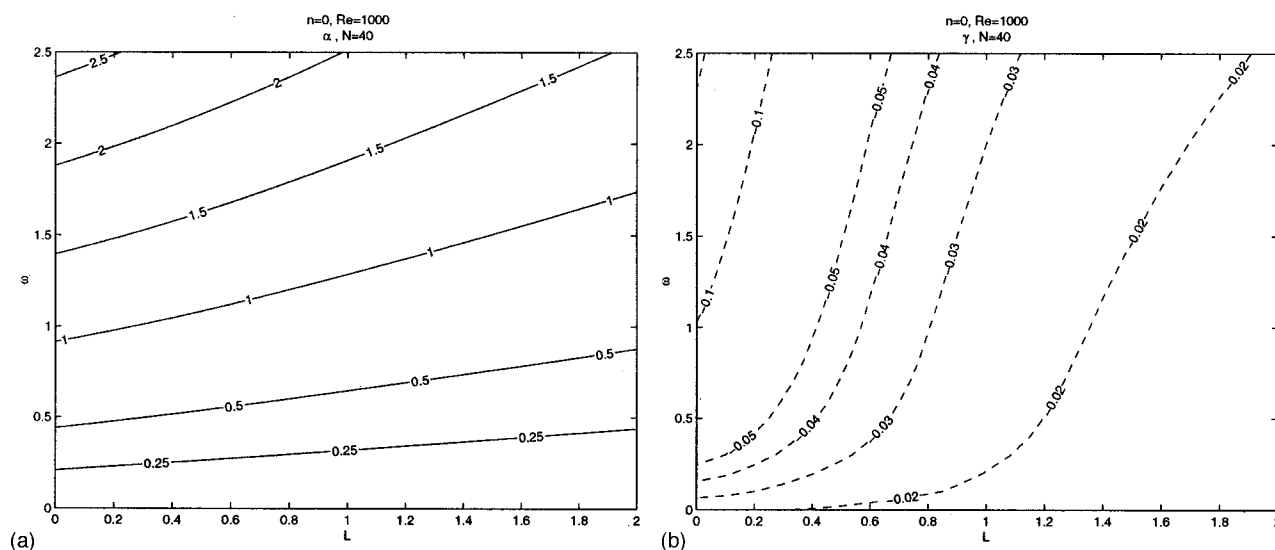


FIG. 1. Contours of constant axial wave number (a) and constant growth rate (b) of the least stable axisymmetric modes ($n=0$) in the (L, ω) -plane for $Re=10^3$ (dashed lines correspond to negative values).

responds to a *convectively* unstable flow, since the growing perturbation is advected downstream of the source with the forcing frequency ω , leaving the flow in its undisturbed state when the forcing ceases.⁹

For axisymmetric perturbations ($n=0$), the flow is spatially stable for every value of the parameters L and Re for any real frequency ω , thus confirming previous temporal stability results. Figure 1 shows the contours of the growth rate and the axial wave number in the (L, ω) -plane of the least stable modes for $n=0$ and $Re=1000$ (only positive frequencies, $\omega>0$, have to be considered for $n=0$ owing to the symmetry property mentioned above). The growth rate γ is always negative, with the least stable perturbations being those with $\omega\rightarrow 0$, which are associated with $\alpha\rightarrow 0$ (large wavelength limit), as L increases. As shown in Fig. 2 for

$Re=1000$ and $\omega=0.01$, the flow tends to be neutrally stable to axisymmetric perturbations as the swirl parameter of the flow is increased indefinitely (which is a known result for inertial perturbations in a purely solid body rotation).²¹

For nonaxisymmetric perturbations ($n\neq 0$), the first azimuthal wave number that becomes spatially unstable as Re , or L , increases is $n=-1$, in agreement with previous temporal stability results. Figure 3 shows the contours of γ and α of the least stable modes (largest γ) in the (L, ω) -plane for $n=-1$, and two values of the Reynolds number, $Re=90$ and $Re=100$. For $Re=90$ [Figs. 3(a) and 3(b)], the flow is unstable ($\gamma>0$) in a narrow strip around the line $L\approx 0.41-0.88\omega$, for $\omega<-0.39$, approximately. The unstable region in Fig. 3(a) lies entirely within the region with $\alpha>0$

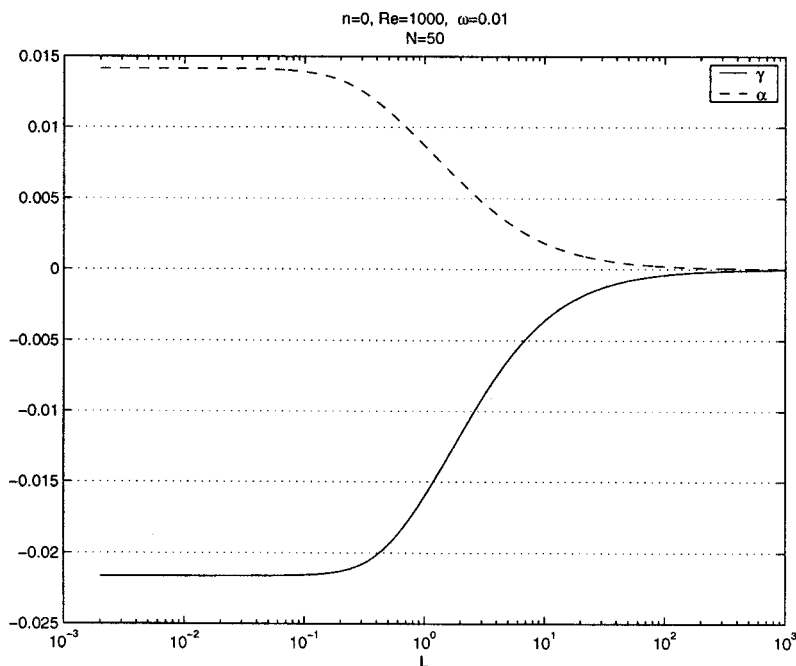


FIG. 2. Spatial growth rate γ and axial wave number α of the least stable mode as functions of L for $n=0$, $Re=10^3$, and $\omega=0.01$.

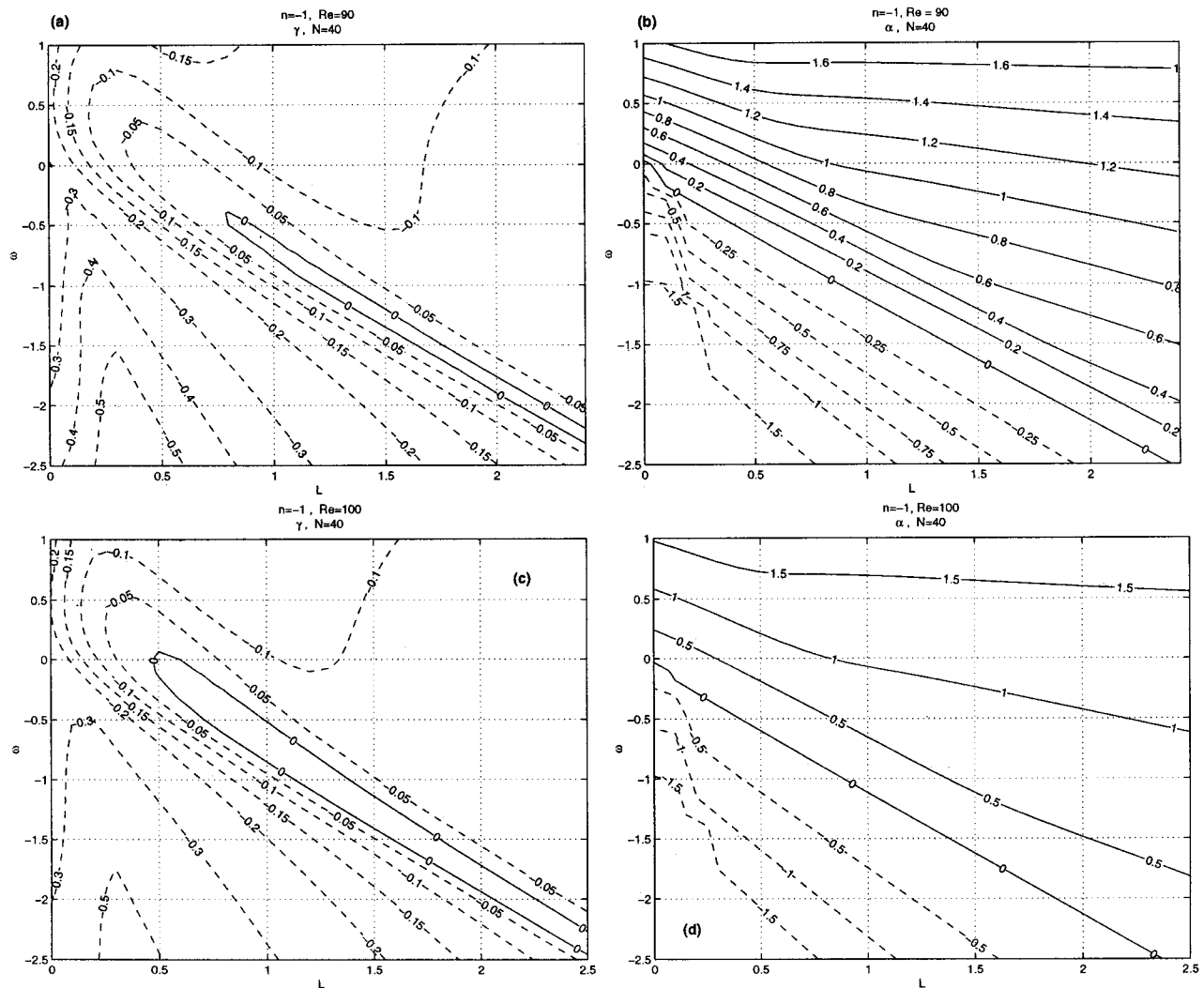


FIG. 3. Contours of constant growth rate [(a) and (c)] and constant axial wave number [(b) and (d)] of the least stable nonaxisymmetric modes with $n = -1$ in the (L, ω) -plane for $\text{Re}=90$ [(a) and (b)] and $\text{Re}=100$ [(c) and (d)]. Dashed lines correspond to negative values.

of Fig. 3(b), so that these unstable perturbations for $\text{Re}=90$ have negative phase velocities,

$$c \equiv \frac{\omega}{\alpha} < 0. \quad (26)$$

However, their real group velocities are positive, as shown in Fig. 4(a) for $L=1$. Thus, although the crests of the unstable waves travel towards decreasing z , the wave packets with the selected frequency ω grow towards increasing z . The results of negative phase speed for these unstable modes with relatively large swirl numbers and low Reynolds numbers are in agreement with the temporal results.^{2,5}

As the Reynolds number decreases below 90, the unstable region in the L, ω plane becomes narrower, and moves towards increasing values of L (decreasing ω), until it vanishes at $\text{Re}=\text{Re}_p \approx 82.9$, with $L \rightarrow \infty$ ($\omega \rightarrow -\infty$). This is the lowest Reynolds number for instability found in previous works on the temporal stability of this flow, first given by Pedley.² On the other hand, as the Reynolds number increases, the instability region in the L, ω plane widens. For $\text{Re}=100$ [Figs. 3(c) and 3(d)], the region with $\gamma > 0$ has al-

ready reached the half-plane $\omega > 0$ [see Fig. 3(c)], implying that now there exist unstable modes with phase speed and group velocity both positive [see Fig. 4(b) for $L=0.5$]. This situation first occurs at $\text{Re} \approx 97.35$ for $L \approx 0.535$.

Figure 5(a) shows the neutral curves (curves on which $\gamma=0$) in the (L, Re) -plane for $n=-1$ and several values of the real frequency ω . As mentioned above, the minimum Reynolds number for instability, $\text{Re}=\text{Re}_p \approx 82.9$, corresponds to the limit $\omega \rightarrow -\infty$ and $L \rightarrow \infty$. It is observed that the instability region for a negative frequency ω becomes narrower as $|\omega|$ increases. The neutral curves for these negative values of ω have the asymptotes $L \rightarrow -\omega$ for $\text{Re} \rightarrow \infty$, with the instability region located to the right of $L = -\omega$ ($L \geq -\omega$). In general, for $\text{Re} \rightarrow \infty$ (inviscid limit), the stability equations have solutions with $\gamma \geq 0$ only if $nL - \omega \geq 0$.

The stability boundary for small L is better appreciated in Fig. 5(b), where the neutral curves are plotted on the $(\text{Re}_\theta = \text{Re}L, \text{Re})$ -plane. The flow is unstable for perturbations with $n=-1$ if $\text{Re}_\theta > 26.96$ (i.e., $\text{Re} > 26.96/L$). This minimum value of the swirl Reynolds number for instability coincides with that obtained by Mackrodt⁴ using a temporal

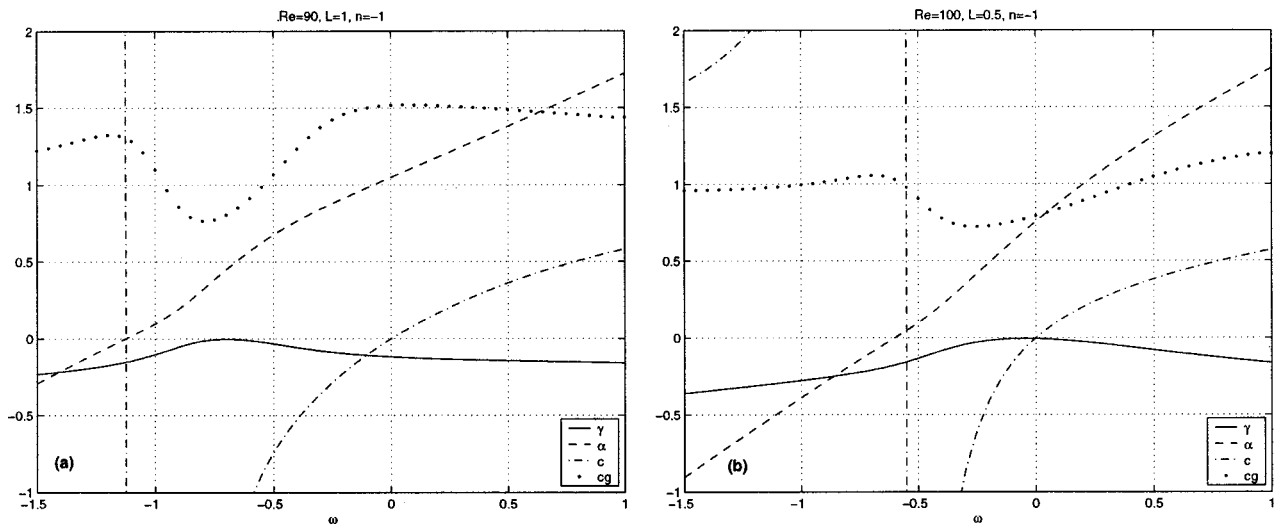


FIG. 4. Spatial growth rate γ , axial wave number α , phase speed c , and real group velocity c_g of the least stable mode as functions of the frequency ω for $n = -1$, $Re=90$, $L=1$ (a), and $n = -1$, $Re=100$, $L=0.5$ (b).

stability analysis. In fact, the envelope of all the neutral curves, which represents the overall stability boundary $Re_c(L)$ (see the next section for more details), obviously coincides with the boundary obtained from the temporal stability analysis.^{4,5}

As $|n|$ increases, the instability region in the (Re, L) -plane becomes smaller, with the stability boundary located at higher values of both Re and L . This can be appreciated in Fig. 6, where the envelopes of the neutral curves for the different frequencies are plotted for $n = -1, -2$ and -3 in both the (L, Re) and the (Re, L) planes. The asymptotic values of the neutral curves for $ReL \rightarrow \infty$ and $Re \rightarrow \infty$ shown in this figure are in agreement with those found by Cotton and Salwen⁵ in their temporal stability analysis.

Although $n = -1$ is the azimuthal wave number that first becomes unstable as Re increases for any given value of L [say at $Re_c(L) \equiv Re_{c1}(L)$], there exists a second Reynolds

number $Re_{c2}(L) > Re_c(L)$ above which some mode with $n = -2$ and a particular frequency becomes more unstable than any mode with $n = -1$. In the same way, for $Re > Re_{c3}(L) > Re_{c2}(L)$, modes with $n = -3$ become the most unstable for a given value of L , and so on. In fact, as it was first shown by Pedley,¹ the most unstable modes in the *inviscid* limit $Re \rightarrow \infty$ correspond to $|n| \rightarrow \infty$ (see also Refs. 3 and 6). In addition, as in the temporal stability analysis discussed in detail by Cotton and Salwen,⁵ we also find in the present spatial analysis several mode switching in the neutral curves for the different frequencies when $n = -2, -3, \dots$. However, all these details are not reported here because our main concern is to characterize the neutral curves for the different values of n , and their corresponding real frequencies.

IV. THE ONSET OF ABSOLUTE INSTABILITY

The neutral curves given in Figs. 5 and 6 mark the onset of convective instability: For a given value of L , when Re is

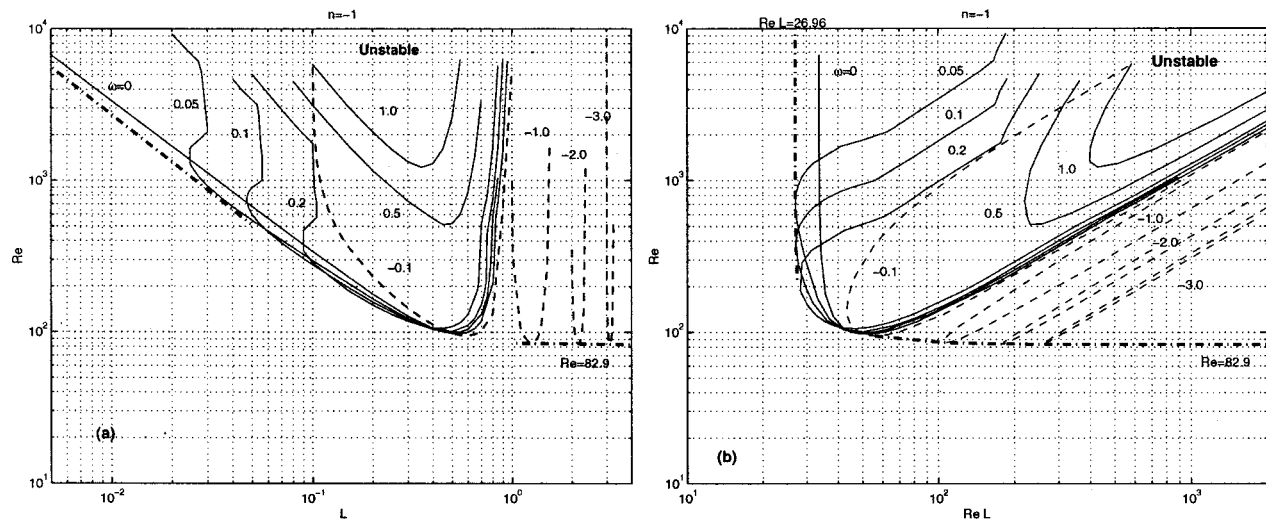


FIG. 5. Neutral curves for $n = -1$ and several frequencies on the L, Re plane (a) and on the Re_θ, Re plane (b). Dashed curves correspond to negative values of ω . The dashed and dotted lines represent the envelopes for both small and large values of the swirl parameter.

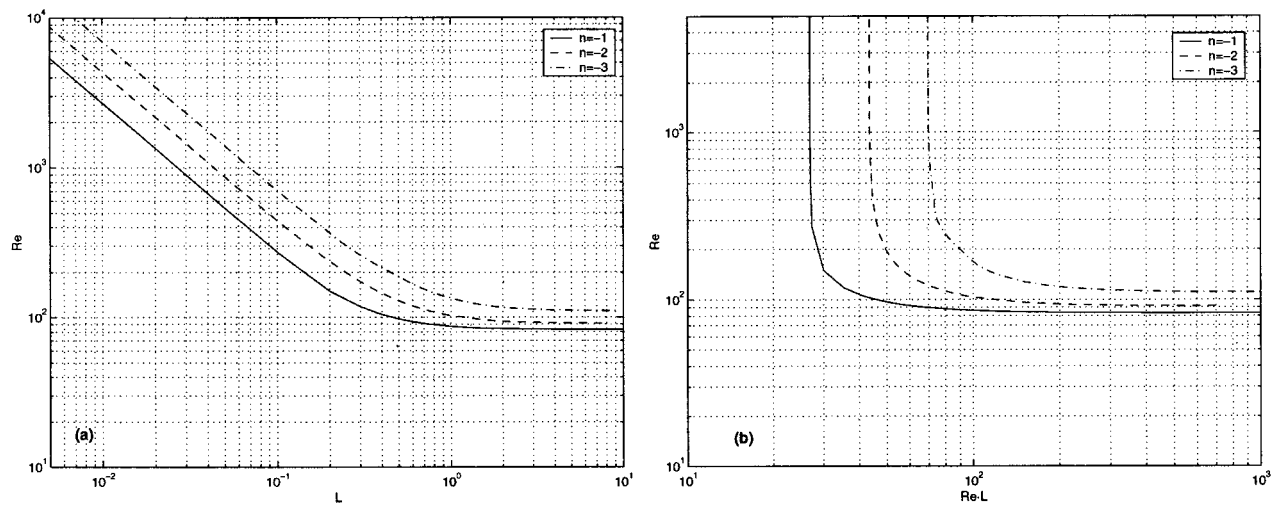


FIG. 6. Envelopes of the neutral curves for convective instability for $n = -1$, $n = -2$, and $n = -3$ on the L, Re plane (a) and on the Re_θ, Re plane (b).

slightly larger than $Re_c(L)$, where $Re_c(L)$ is given by the envelope of the neutral curves in Fig. 5 for $n = -1$, the growth rate γ becomes positive for at least some frequency ω . The unstable wave packet with that frequency travels at a positive real group velocity c_g (see Fig. 4), so that it corresponds to a convective instability. The same can be said for the perturbations with $n = -2$ and $n = -3$ when Re becomes larger than the corresponding neutral curves plotted in Fig. 6.

It is observed in Fig. 4 that c_g has its minimum value, for given Re and L , at the same frequency, approximately, at which γ is maximum. That is to say, wave packets corresponding to the most unstable (or least stable) modes are the slowest traveling along the pipe. As the Reynolds number is increased, the maximum growth rate for a given L increases,

and the corresponding minimum of c_g decreases until it vanishes in a cusp point when $Re = Re_c(L) > Re_c$ at a frequency $\omega_c(L)$ [see Figs. 7 and 8 for two different values of L and $n = -1$; in these figures, like in Fig. 4, c_g [Eq. (25)] is numerically computed using second-order centered differences of $\alpha(\omega)$, taking $\Delta\omega$ as small as 10^{-4} in the vicinity of the cusp points]. For these values of the frequency and Reynolds number, the growth rate presents also a cusp point [see Figs. 7(a) and 8(a)], so that the flow is markedly more unstable at that frequency. But, what is more important, two spatial branches of the dispersion relation coalesce at the real frequency $\omega = \omega_c(L)$ when $Re = Re_c(L)$, indicating that the flow may become absolutely unstable according to the Briggs–

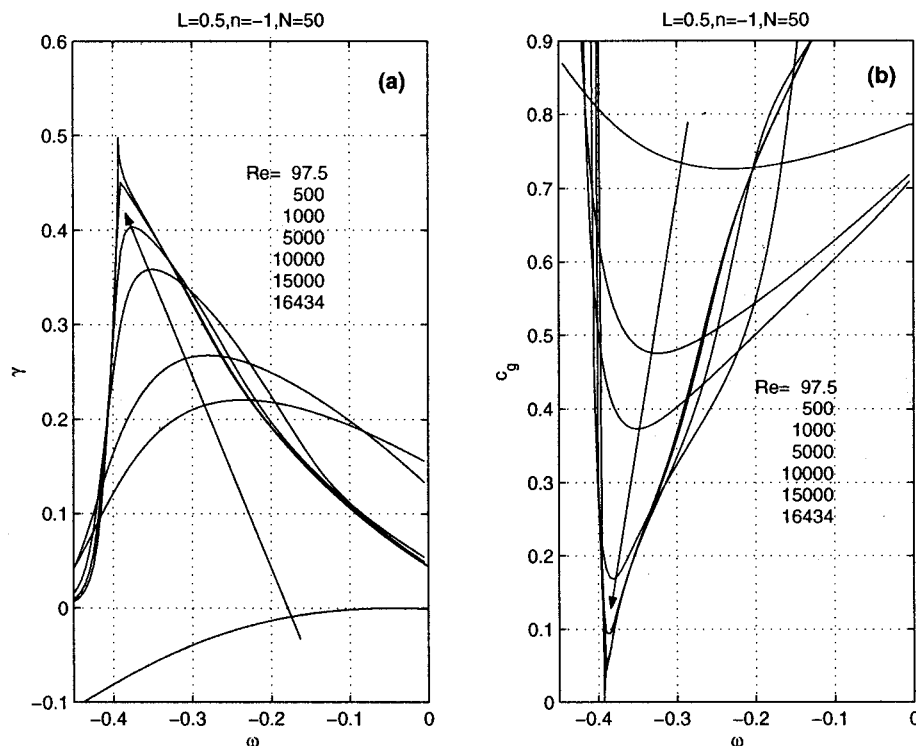


FIG. 7. Spatial growth rate γ (a) and real group velocity c_g (b) vs frequency for $n = -1$, $L = 0.5$ and several Reynolds numbers between $Re_c(0.5) \approx 97.5$ and $Re_c(0.5) \approx 16434$.

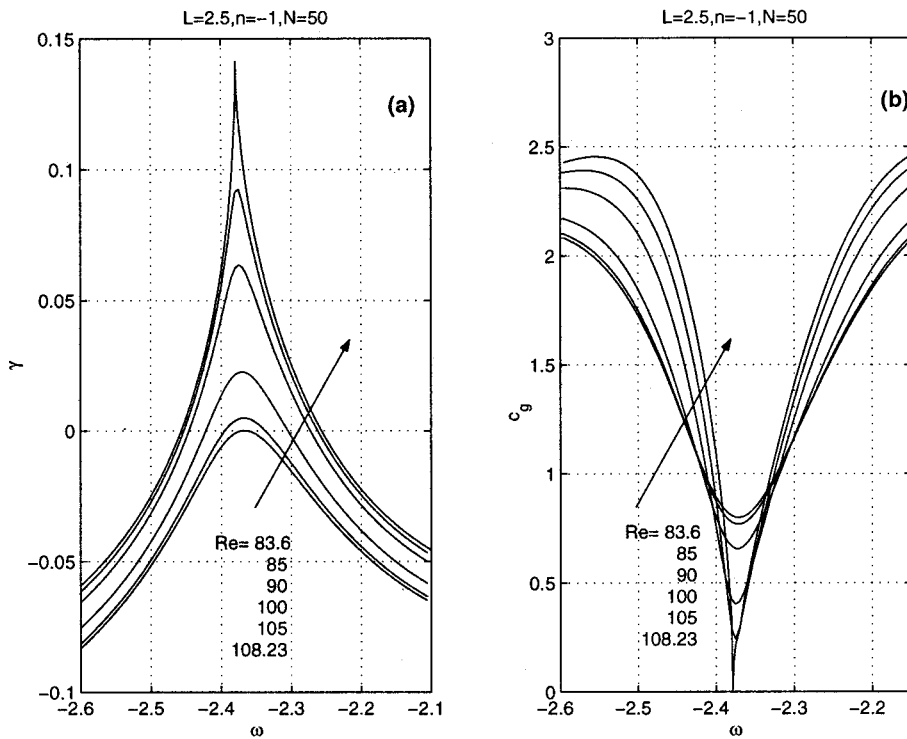


FIG. 8. As in Fig. 7, but for $L=2.5$. $\text{Re}_c(2.5) \approx 83.6$, $\text{Re}_t(2.5) \approx 108.23$.

Bers criterion. Actually, if one defines the complex group velocity (see, e.g., Refs. 10 and 11)

$$v_g \equiv \frac{d\varpi}{dk} = i \frac{d\varpi}{da} = \frac{\partial \omega}{\partial \alpha} + i \frac{\partial \omega}{\partial \gamma} = c_g + i \frac{\partial \omega}{\partial \gamma}, \quad (27)$$

where $\varpi \equiv \omega + i\omega_i$ is the complex frequency and $k \equiv a/i = \alpha - i\gamma$ is the complex axial wave number [see Eqs. (13) and (25)], v_g vanishes at $\text{Re}_t(L)$ when $\omega = \omega_t(L)$ because $c_g = 0$ and $\partial \gamma / \partial \omega \rightarrow \infty$. In addition, the cusp points in Figs. 7 and 8 are *pinching* points for two spatial branches k^+ and k^- in the complex (α, γ) -plane, the *positive* branch retreating to the lower half ($\gamma < 0$) of the complex plane as $\omega_i = \Im(\varpi)$ increases, while the *negative* branch lies entirely in the upper

half of the complex (α, γ) -plane as ω_i increases (see Fig. 9 for the same values of n and L considered in Figs. 7 and 8). These two conditions characterize the transition to absolute instability according to the Briggs-Bers criterion (see, for instance, Ref. 10). For $\text{Re} > \text{Re}_t(L)$, both spatial branches become *mixed*, and the present spatial stability analysis is no longer appropriate. However, Figs. 7, 8, and 9 show that the present spatial analysis is an efficient tool to search for the onset of absolute instability in this kind of flow: one only has to look for conditions at which $c_g = 0$ in a cusp point. Then, if they exist, check whether they correspond to a pinching point of two spatial branches (positive and negative) in the complex wave number plane.

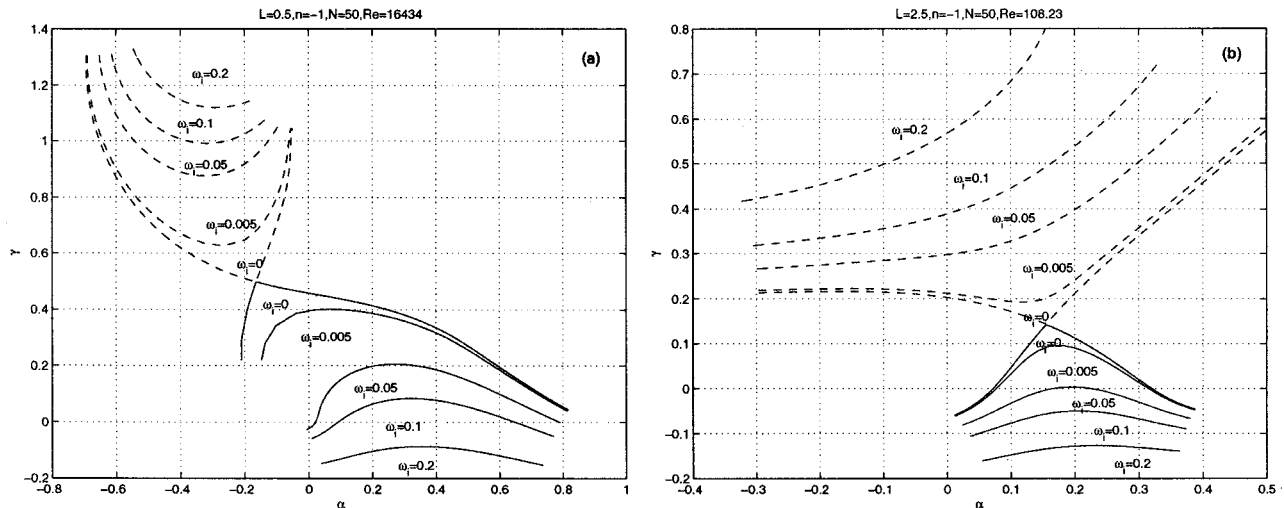


FIG. 9. Spatial branches k^+ (solid lines) and k^- (dashed lines) in the complex (α, γ) -plane for different values of ω_i at the conditions corresponding to the cusp points of Fig. 7(a), and Fig. 8(b). Note that the complex axial wave number is, in the present notation, $k = a/i = \alpha - i\gamma$.

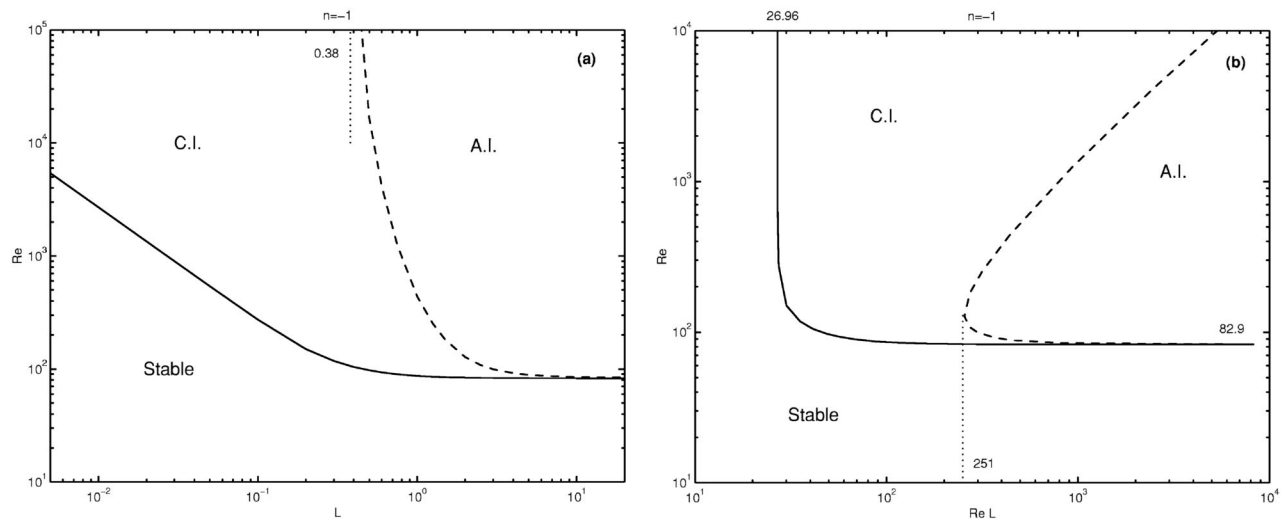


FIG. 10. Regions of stability, convective instability (CI), and absolute instability (AI) for $n = -1$ in the (L, Re) -plane (a), and in the $(\text{Re } L, \text{Re})$ -plane (b). These regions are separated by the curves $\text{Re}_c(L)$ (solid) and $\text{Re}_i(L)$ (dashed). The dotted lines mark the minimum swirl for absolute instability.

At the conditions of the cusp points of Figs. 7 and 8 [i.e., at $\text{Re} = \text{Re}_i(L)$] perturbations with $n = -1$ and the real frequency $\omega = \omega_i(L)$ are unstable (γ is very large indeed) and have zero complex group velocity, so that they grow *in situ*. For $\text{Re} > \text{Re}_i(L)$, there exist unstable perturbations that propagate upstream, but they are no longer described by the present spatial stability analysis. Figure 10 shows the boundary $\text{Re}_i(L)$ together with $\text{Re}_c(L)$ for $n = -1$, which is the azimuthal wave number that first becomes unstable, both convectively and absolutely, as the Reynolds number is increased for each L (see below). An interesting feature is that the flow can be absolutely unstable only if the swirl parameter L is above a critical value $L^* \approx 0.38$ (see also Fig. 11), while the flow may be convectively unstable for any value of L if the Reynolds number is high enough. In terms of the swirl Reynolds number [Fig. 10(b)], the flow can be convectively unstable for $\text{Re } L > 26.96$, whereas it can be absolutely unstable if $\text{Re } L > 251$, approximately. For large L ($L > 20$,

approximately), $\text{Re}_i(L) \approx \text{Re}_c(L)$, and the flow becomes absolutely unstable just after becoming convectively unstable, as Re is increased for a given L .

Similar computations have been carried out for $n = -2, -3, \dots$. Figure 11 compares the boundaries of absolute instability for $n = -1, -2$, and -3 , showing that $n = -1$ is the azimuthal wave number that first becomes absolutely unstable for each L as Re is increased. Higher values of $|n|$ are not shown because they follow a similar trend. It is observed that perturbations with $n = -2$ can be absolutely unstable only if $L > L_2^* \approx 0.55$, while for perturbations with $n = -3$, L must be larger than 0.64 (approximately) to become absolutely unstable as Re is increased. As in the case $n = -1$, the boundaries for convective and absolute instability, $\text{Re}_c^{(n)}(L)$ and $\text{Re}_i^{(n)}(L)$, practically coincide for large L for all the values of n considered (compare Fig. 11 with Fig. 6).

Figure 12 shows the frequencies $\omega_c^{(n)}(L)$ and $\omega_i^{(n)}(L)$

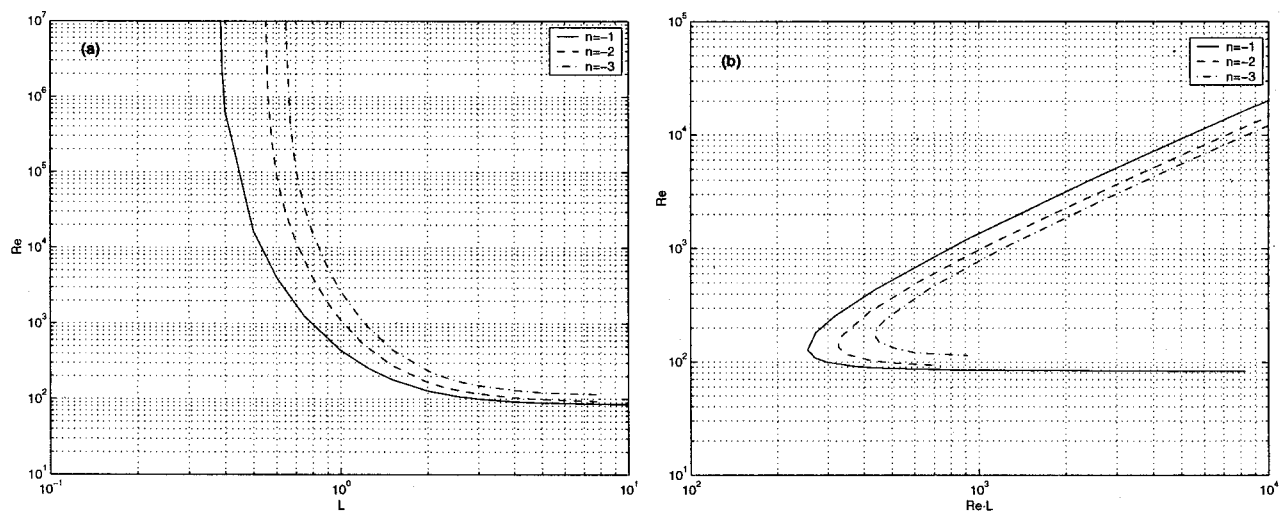
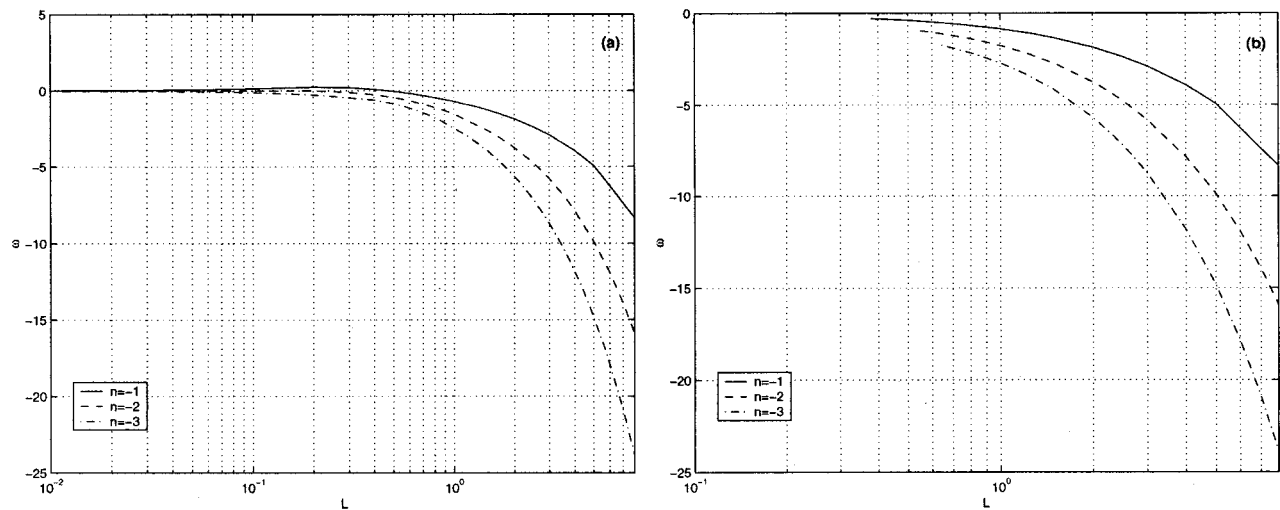


FIG. 11. Boundaries of absolute instability for $n = -1, -2$, and -3 .

FIG. 12. $\omega_c^{(n)}(L)$ (a) and $\omega_t^{(n)}(L)$ (b) for $n = -1, -2$, and -3 .

for the onset of convective and absolute instabilities, respectively, for perturbations with $n = -1, -2$, and -3 . For each n , these are the frequencies that first become unstable as Re is increased for a given L . For large L , these two frequencies practically coincide as a consequence of the fact that $Re_c^{(n)}(L) \approx Re_t^{(n)}(L)$ for large L . They can be approximated by $\omega_t^{(n)} \approx \omega_c^{(n)} \approx nL$, $L \gg 1$. Note that $\omega_t^{(n)}(L)$ only exists for $L > L_n^*$, where $L_n^* \approx 0.38, 0.55$, and 0.64 for $n = -1, -2$, and -3 , respectively.

Finally, Fig. 13 shows the amplitude of the eigenfunctions for $n = -1$ and $L = 0.5$ corresponding to two modes, one on the neutral curve for convective instability, $Re = Re_c(0.5) = 97.5$ and $\omega = \omega_c(0.5) = -0.04$, and the second one at the onset of absolute instability, $Re = Re_t(0.5) = 16434$ and $\omega = \omega_t(0.5) \approx -0.39$. For a relatively low Reynolds number such as $Re = 97.5$ [Fig. 13(a)], the disturbances are *center* modes, with the maxima of the disturbance velocities at, or near, the pipe axis. However, for large Reynolds numbers [Fig. 13(b)], the most unstable modes are *wall* modes.

V. CONCLUDING REMARKS

The main results of the spatial stability analysis of rotating Hagen–Poiseuille flow performed in this work are summarized in Fig. 10 (together with Figs. 6 and 11). The analysis corroborates the stability boundary (for convective instabilities) obtained previously from the temporal stability of the flow.^{4,5} The spatial analysis yields the neutral curves for each real frequency of the disturbances (e.g., Fig. 5), instead of their wave number, which is a more relevant information from an experimental point of view. In addition, the spatial analysis provides a relatively simple tool for finding out the absolute–convective transition as the Reynolds number is increased for a given swirl number. In particular, it is shown that wave packets corresponding to the most unstable frequency for each Re and L are the slowest propagating downstream, i.e., they have the smallest real group velocity c_g . As Re is increased, the complex group velocity v_g eventually vanishes at $Re_t(L)$, and the flow becomes absolutely unstable according to the Briggs–Bers zero-group-

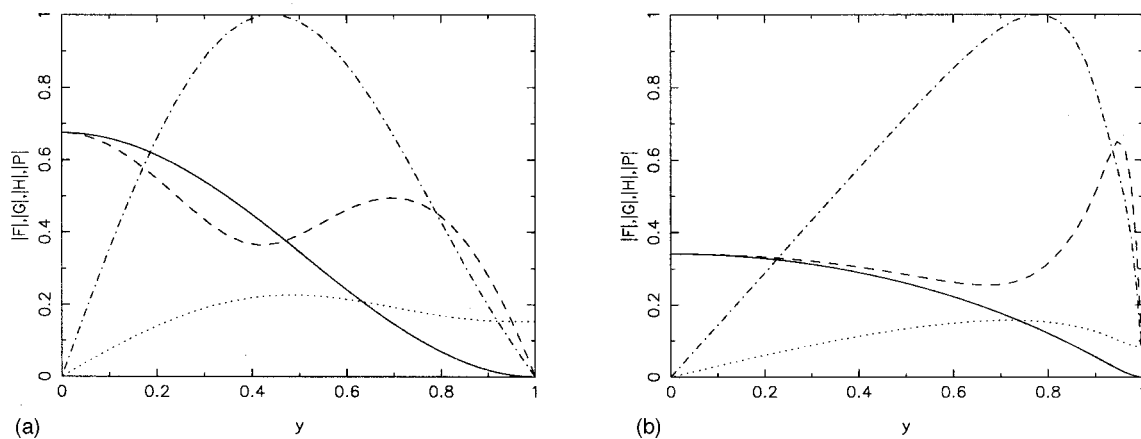


FIG. 13. Eigenfunctions amplitudes $|F|$ (solid lines), $|G|$ (dashed lines), $|H|$ (dashed and dotted lines), and $|P|$ (dotted lines) for $n = -1$, $L = 0.5$ with $Re = Re_c(0.5) = 97.5$ and $\omega = \omega_c(0.5) = -0.040$ (a), and $Re = Re_t(0.5) = 16434$ and $\omega = \omega_t(0.5) = -0.3929$ (b). The maximum value of $|H|$ is normalized to unity.

velocity criterion. It is remarkable that the flow can be absolutely unstable only for $L > L^* \approx 0.38$ [$\text{Re}_\ell(L^*) \rightarrow \infty$ for $n = -1$].

The present stability analysis is similar to that given by Olendraru *et al.*¹⁴ for the Batchelor vortex. However, the onset of absolute instability is searched here in a more straightforward way by first looking for the conditions at which the real part of the complex group velocity vanishes. The main qualitative difference between the present results and those for the Batchelor and related vortices^{11–16} is that the rotating Hagen–Poiseuille flow may be absolutely unstable for relatively low values of the swirl parameter in spite of the absence of wake-like axial velocity profile. In addition, it is shown that perturbations with the azimuthal wave number $n = -1$ are the first to become absolutely unstable as both the Reynolds number is increased, for every $L > L^*$, and the swirl number is increased for every $\text{Re} > \text{Re}_p \approx 82.9$. As a final comment, in the references just cited, the onset of absolute instability in Batchelor, Rankine, and related vortices is associated with the vortex breakdown phenomenon. The relation between stability and vortex breakdown is also argued for an inviscid flow with solid body rotation in a pipe of finite length by Wang and Rusak,^{22,23} who analyzed the inviscid stability for axisymmetric perturbations. Unfortunately, we cannot speculate here on the nature of the present rotating pipe flow after becoming absolutely unstable because no experimental data are available.

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